1968

A lower bound solution for drawing of wide strip through inclined planes in plane strain

Fred R. Sauerwine
Lehigh University

A LOWER BOUND SOLUTION FOR DRAWING OF WIDE STRIP THROUGH INCLINED PLANES IN PLANE STRAIN

by

Fred R. Sauerwine

A THESIS

Presented to the Graduate Faculty

of Lehigh University

in Candidacy for the Degree of

Master of Science

in

Metallurgy and Materials Science

Lehigh University

1968

CERTIFICATE OF APPROVAL

This thesis is accepted and approved in partial fulfillment of the requirements for the degree of Master of Science.

APRIL 1, 1968

Betzalel Avitzur
Professor in charge

Head of the Department

ACKNOWLEDGMENTS

Dr. Betzalel Avitzur is thanked for the many valuable suggestions which he made throughout the course of this work. The author also wishes to thank Mr. Jose Fueyo for his help during the initial stages of this solution.

The financial support of the United States Department of Health, Education, and Welfare is acknowledged.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

ABSTRACT

A lower bound solution is attempted for the process of drawing wide strip through inclined planes in plane strain. Since a solution becomes possible if the deforming body is separated into several regions, the wide strip is first divided into five zones: the undeformed zone, the three zones of deformation, and the zone of completed deformation. Across the boundary between each zone, stress discontinuity is permitted without disturbing the equilibrium. To find boundary conditions needed later in the determination of the stress fields in the deformation zones, statically admissible stress fields are assumed in the undeformed and the completely deformed regions, and friction is described by the constant shear factor assumption. The three obtained deformation region stress fields are represented by smooth functions (The functions and their derivatives are continuous). Also, these stress fields in the deforming zones obey the equilibrium equations, the von Mises yield criterion with assumed yielding at ($\Theta = 0^\circ$), and the imposed boundary conditions. However, to determine a lower bound solution for the drawing stress (σ_{xx_I}), the von Mises yield criterion must be satisfied for all values of the coordinates (r) and (Θ). Therefore, a lower bound value for the drawing stress (σ_{xx_I}) allows the von Mises yield criterion to be satis-

fied throughout each zone when this yield criterion is evaluated at practical values for the independent process variables (semi-cone angle, shear factor, reduction, and back-pull). To be useful, the drawing stress (σ_{xx_I}) must be located between the "ideal" solution and an upper bound solution.

Using the previously suggested approach gives no practical value for the drawing stress (σ_{xx_I}). In fact, the drawing stresses found from this solution appear to be considerably lower than the "ideal" values.

By assuming a distribution of the normal stress (σ_{rr}) at ($\theta = 0°$) instead of satisfying the von Mises yield criterion with assumed yielding at ($\theta = 0°$), a different lower bound solution is found. This lower bound solution yields drawing stress values which are also below the "ideal" values, but higher than the drawing stresses obtained in the initial approach.

Suggestions are made for future attempts to determine a practical lower bound solution.

INTRODUCTION

In the metal forming process of wide strip drawing, the thickness (t) of the strip is reduced from (t_o) to (t_f) by drawing it through two inclined planes with an included angle of (2α). (Refer to Figure 1.) Since the width of the strip is much larger than the thickness, the assumption is made that no width changes occur during deformation. (The strains in the Z - direction (ε_{iz}) are zero.) However, the principal stress in the direction of the (Z) coordinate (σ_{zz}) is non-zero and is a function of only the (r) and (θ) coordinates. (Refer to equation (h) in Appendix I.) With these conditions, a plane strain problem exists.

Depending upon the assumptions made, many lower bound solutions are possible. If any restriction is imposed on the problem, the lower bound solution becomes excessively lower; the confining assumption of plane strain has this effect. The lower bound solution is decreased also by assuming a certain stress distribution to solve the equilibrium equations. However, since assumptions allow solutions to become possible or easier to obtain, they are created with the hope that the lower bound solution is still practical.

A lower bound solution is termed practical if it is less than an upper bound solution and greater than the "ideal" solution. The actual externally supplied power is neither higher than the power computed by using the upper bound theorem of Prager and Hodge (equation [37.2]; reference [17]), nor less than the power computed by using the lower bound theorem of Prager and Hodge (equation [37.1]; reference [17]). Also, the actual power is never lower than the "ideal" power of deformation. However, the "ideal" power is usually too low to be useful because friction and distortion are neglected completely and only reduction is considered. Therefore, this work is concerned with finding a lower bound solution located between the upper bound and the "ideal" solutions.

A method used to obtain lower bound solutions for plane strain problems involves slip line field theory. The slip line technique has been used by Alexander (1) and by Ellis (8) to determine lower bound solutions for frictionless, plane-strain extrusion. Also, Richmond and Devenpeck (18) have used slip line field theory to calculate an ideal, strip drawing die profile which they later evaluated through experimentation. (5)

Another approach for determining a lower bound solution is suggested by Avitzur in chapter 6 of reference (2). He recommends dividing the deforming body into several zones and then determining a statically admissible stress field in each zone. Without disturbing the equilibrium, stress field discontinuities are permitted across the boundaries between two neighboring regions.

Therefore, the purpose of this work is to determine a practical lower bound solution for the forming process of wide strip drawing in plane strain by using the suggested approach of Avitzur.

THE LOWER BOUND APPROACH

Lower Bound Theorem

According to Prager and Hodge (17), the lower bound theorem is "Among all statically admissible stress fields, the actual one maximizes the expression"

$$I = \int_{S_v} T_i v_i \, ds \tag{1}$$

where (I) is the computed power supplied by the tool which is applied to surfaces having prescribed velocity, (T_i) are the normal traction components acting on surfaces with a prescribed velocity, and (v_i) is the velocity of the tool. (2)

The lower bound theorem (equation [1]) is an extremum principle concerning a body composed of a material obeying von Mises' stress-strain rate law

$$\dot{\varepsilon}_{ij} = \mu S_{ij} \tag{2}$$

where ($\dot{\varepsilon}_{ij}$) is the strain rate tensor, (μ) is a scalar function of the strain rates, and (S_{ij}) is the stress deviator tensor. Also, the von Mises yield criterion (16) must be satisfied. Expressed in cylindrical coordinates, the von Mises yield criterion is

$$\frac{1}{6}\left[(\sigma_{rr} - \sigma_{\theta\theta})^2 + (\sigma_{\theta\theta} - \sigma_{zz})^2 + (\sigma_{zz} - \sigma_{rr})^2 \right] + \sigma_{r\theta}^2 + \sigma_{\theta z}^2 + \sigma_{zr}^2 \leq \sigma_0^2/3 . \tag{3}$$

At the stress (σ_o), a material under uniaxial tension yields. When the left side of the yield criterion is less than ($\sigma_o^2/3$), the material is in the elastic range, and when the left side is equal to ($\sigma_o^2/3$), yielding is initiated.

For the plane strain problem of wide strip drawing, the normal traction component is the drawing stress (σ_{xxI}), the final velocity of the wide strip (v_f) is equal to the velocity (v_i) of the tool pulling the strip through the die, and the surface on which the drawing stress acts is the product of the final thickness of the wide strip (t_f) and the width (unity). (Refer to Figure 1.) Therefore, the lower bound on power (equation [1]) becomes

$$I = \int_{S_v} (\sigma_{xx_I})(v_f)\, ds = \sigma_{xx_I} \cdot v_f \cdot t_f \, . \tag{4}$$

As the power (I) increases, the statically admissible stress field used to obtain (I) is presumed to be closer to the actual stress field. Even if the statically admissible stress field is not close to the actual one, (I) approaches the exact value quite rapidly. (2) Therefore, maximizing the value of (I) is desirable. From equation (4), (I) can be maximized if (σ_{xxI}) is a maximum.

Statically Admissible Stress Field

To apply the lower bound theorem (equation [1]), the stress fields must be statically admissible. A stress field is statically admissible if the following conditions are fulfilled:

(1) The stress field should be a smooth function. (The function and its derivatives must be continuous except over surfaces of stress discontinuity [s]).

(2) The equations of equilibrium must be obeyed.

(3) The imposed static boundary conditions need to be satisfied.

(4) The von Mises yield criterion (equation [3]) must not be violated at any point in the body.

A solution becomes possible or is made simpler by dividing the wide strip into several zones. In each zone, the stress state is described by statically admissible stress fields. However, between each zone, surfaces of stress discontinuity (s) are allowed without upsetting the equilibrium.

Stress Field Discontinuity

Prager and Hodge describe discontinuous stresses in reference (17), p. 155. Across the surface of stress discontinuity, the shear components are continuous, and the stresses normal to the surface are equal. The discontinuity occurs in the normal stresses acting on the normal plane through the surface.

The surface of stress discontinuity (s_1) is illustrated in Figure 2. The shear stresses (T_I) and ($\sigma_{r\theta}^{II}$) at the boundary are equal;

$$T_I = \sigma_{r\theta}^{II} \big|_{r=r_f} \qquad (5)$$

the normal stresses (N_I) and (σ_{rr}^{II}) are also equal at $(r = r_f)$.

$$N_I = \sigma_{rr}^{II}\big|_{r=r_f} \tag{6}$$

However, the normal stresses $(\sigma_{\theta\theta}^{I}/_{r=r_f})$ and $(\sigma_{\theta\theta}^{II}/_{r=r_f})$ are unequal. For wide strip drawing in plane strain,

$$\sigma_{\theta\theta}^{I}\big|_{r=r_f} = 0; (\text{ assumed equal to zero }) \tag{7}$$

$$\sigma_{\theta\theta}^{II}\big|_{r=r_f} < 0.$$

<u>Shear and Friction</u>

The friction resistance occurring at the die-wide strip interface is equal to the shear stress (τ) in the wide strip at this boundary. Two mathematical approaches are considered for describing friction: the Coulomb coefficient of friction and the constant shear factor approaches.

According to Coulomb,

$$\tau = \mu p \tag{8}$$

where (μ) is the Coulomb coefficient of friction. (μ) is constant for a certain die and material under constant surface and temperature conditions; it is also independent of velocity. (2) This friction coefficient has been experimentally determined by Lancaster and Rowe (15) and Fukui, <u>et. al.</u> (10) for several materials being drawn under plane strain conditions. The other description assumes (τ) to be a constant independent of the pressure between the die and the wide strip.

The constant shear factor approach is

$$\tau = m \frac{\sigma_o}{\sqrt{3}} \tag{9}$$

where (m) is the shear factor, (σ_o) is the yield stress of the material under uniaxial tension, and ($\sigma_o/\sqrt{3}$) is the maximum shear stress that a material can withstand according to the von Mises yield criterion in equation (3). The shear factor (m) is considered constant for a certain die and material under constant surface and temperature conditions; it is also independent of velocity. (2) The values of (m) vary between zero (frictionless) and one (dead zone formation). Using the rod drawing process, (μ) and (m) have been determined experimentally for very low carbon ($<.02\%$C) and medium carbon (1024) steel by Evans and Avitzur (9).

Since the purpose of this work is to determine a practical lower bound solution, the "ideal" solution and an upper bound solution are required for strip drawing in plane strain. These solutions have been found by Avitzur, et. al. in reference (3). In their analysis, friction is described by the constant shear factor approach because the upper bound theorem (17) applies only if constant shear is assumed. Therefore, for comparison purposes, friction is assumed to obey equation (9) in this lower bound solution.

Approach of Avitzur (2)

The wide strip is first divided into five zones: the undeformed zone V, the three zones in the region of deformation (II, III and IV), and the completely deformed zone I. (Refer to Figure 1.) Statically admissible stress fields are then assumed in zones I and V so that known boundary conditions exist along the permitted boundaries of stress discontinuity (s_1) and (s_4). The stress fields determined in zones II, III and IV are represented by smooth functions, and they satisfy the equilibrium equations, the von Mises yield criterion with assumed yielding at ($\theta = 0^\circ$), and the imposed boundary conditions. A lower bound solution for the drawing stress (σ_{xx_I}) becomes possible if the stress fields in the deformation region obey the von Mises yield criterion throughout each zone.

The solution does not consist of an equation relating the drawing stress (σ_{xx_I}) to the process variables (reduction $\left(\frac{t_o}{t_f}\right)$, semi-cone angle ($\alpha$), back pull ($\sigma_{xx_V}$), and shear factor ($m$)). The drawing stress is the maximum stress satisfying the following two requirements:

(1) (σ_{xx_I}) is located between the "ideal" and the upper bound values for the drawing stress found by Avitzur, Fueyo, and Thompson (3).

(2) (σ_{xx_I}) allows the von Mises yield criterion, evaluated at

specific values of the process variables, to be obeyed throughout each zone.

A LOWER BOUND SOLUTION

Statically admissible stress fields for wide strip drawing are assumed in zones I and V. (Refer to Figure 1.) The wide strip is divided into five zones, each of these zones being separated from the neighboring zones by a cylindrical surface of stress discontinuity (s).

In zone I, the drawing stress (σ_{xxI}) is induced in order to pull the wide strip through two inclined planes which have an included angle of (2α). Zones II, III and IV are the regions of deformation in which the thickness (t) is reduced from (t_o) to (t_f). The back pull (σ_{xxV}) acts in zone V where the wide strip is still undeformed.

Using the assumed statically admissible stress field in zone I, the derivation begins by determining the boundary conditions at the surface of stress discontinuity (s_1) at ($r = r_f$). (Refer to Figure 2.)

The boundary conditions at (s_1) are found by resolving the drawing stress (σ_{xxI}) into its three components (σ_1), (σ_2) and (σ_3) which are parallel to the coordinate axes (x_1), (x_2) and (x_3), respectively, and then by resolving these three components into a normal component (N_I) and a tangential or shear component (T_I).

The nine components of stress in zone I are

$$\sigma_{11} = \sigma_{xx_I}\,; \quad \sigma_{22} = \sigma_{12} = \sigma_{23} = \sigma_{31} = 0$$
$$\sigma_{33} = \frac{1}{2}\left(\sigma_{11} + \sigma_{22}\right) = \frac{1}{2}\sigma_{11} = \frac{1}{2}\sigma_{xx_I}\,. \tag{10}$$

[(σ_{33}) is found by deriving equation (h) in Appendix I for the rectangular coordinate system (x_1, x_2, x_3) instead of the cylindrical coordinate system (r, θ, z).] The directional cosines on surface (s_1) are

$$\begin{aligned} \alpha_1 &= \cos(N, x_1) = \cos\theta \\ \alpha_2 &= \cos(N, x_2) = \cos(90 - \theta) = \sin\theta \\ \alpha_3 &= \cos(N, x_3) = \cos 90^\circ = 0 \,. \end{aligned} \tag{11}$$

(σ_1), (σ_2) and (σ_3) are found by substituting equations (10) and (11) into the following equations (12).

$$\begin{aligned} \sigma_1 &= \sigma_{11}\alpha_1 + \sigma_{21}\alpha_2 + \sigma_{31}\alpha_3 \\ \sigma_2 &= \sigma_{12}\alpha_1 + \sigma_{22}\alpha_2 + \sigma_{32}\alpha_3 \\ \sigma_3 &= \sigma_{13}\alpha_1 + \sigma_{23}\alpha_2 + \sigma_{33}\alpha_3 \end{aligned} \tag{12}$$

Therefore,

$$\begin{aligned} \sigma_1 &= \sigma_{xx_I}\cos\theta \\ \sigma_2 &= 0 \\ \sigma_3 &= 0. \end{aligned} \tag{13}$$

The normal component (N_I) and the shear component (T_I) can now be determined by using the following equations.

$$N = \sigma_1\alpha_1 + \sigma_2\alpha_2 + \sigma_3\alpha_3 \tag{14}$$

$$T = \left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - N^2\right]^{\frac{1}{2}} \tag{15}$$

Substituting equations (11) and (13) into equation (14) gives

$$N_I = +\sigma_{xx_I}\cos^2\theta \tag{16}$$

(By the sign convention (2), (N_I) is positive because it is directed outward from the surface, i.e., a tensile stress.) The shear stress (T_I) is determined by substituting equations (13) and (16) into equation (15).

$$T_I = -\frac{1}{2}\sigma_{xx_I}\sin 2\theta \tag{17}$$

[(T_I) is negative because of the sign convention. Since (N_I) is positive (tensile) and is directed in the negative (r) direction, the shear stress pointing in the positive (θ) direction is negative.]

At the surface of stress discontinuity (s_1) (Refer to Figure 2.), (T_I) is equal to the shear stress ($\sigma^{II}_{r\theta}$) at ($r = r_f$),

$$T_I = \sigma^{II}_{r\theta}/_{r = r_f} \tag{5}$$

and also at (s_1), equilibrium requires that (N_I) be equal to the normal stress (σ^{II}_{rr}) at ($r = r_f$).

$$N_I = \sigma^{II}_{rr}/_{r = r_f} \tag{6}$$

Therefore, the boundary conditions at ($r = r_f$) are

$$\left.\begin{aligned} \sigma^{II}_{r\theta}\big/_{r=r_f} &= -\frac{1}{2}\sigma_{xx_I}\sin 2\theta \\ \sigma^{II}_{rr}\big/_{r=r_f} &= +\sigma_{xx_I}\cos^2\theta \end{aligned}\right\} \tag{18}$$

The stress field in zone II must satisfy the boundary conditions in equation (18) as well as the equilibrium equations.

The equilibrium equations in cylindrical coordinates are

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{rz}}{\partial z} + \left(\frac{\sigma_{rr} - \sigma_{\theta\theta}}{r}\right) = 0$$

$$\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{\theta z}}{\partial z} + 2\frac{\sigma_{r\theta}}{r} = 0 \tag{19}$$

$$\frac{\partial \sigma_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta z}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r} = 0.$$

However, according to Unksov (21), for plane strain with a cylindrical coordinate system

$$\frac{\partial \sigma_{zz}}{\partial z} = 0; \quad \sigma_{rz} = \sigma_{\theta z} = 0. \tag{20}$$

Therefore, the equilibrium equations in equations (19) become

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \left(\frac{\sigma_{rr} - \sigma_{\theta\theta}}{r}\right) = 0$$

$$\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + 2\frac{\sigma_{r\theta}}{r} = 0. \tag{21}$$

If a distribution of the shear stress $(\sigma_{r\theta}^{II})$ is assumed, the two equilibrium equations [equations (21)] have the two unknown stresses (σ_{rr}^{II}) and $(\sigma_{\theta\theta}^{II})$; therefore, these equations can be solved. The shear stress $(\sigma_{r\theta}^{II})$ must obey the assumed distribution of shear and the boundary conditions in zone II.

Since the shear $(\sigma_{r\theta})$ in the upper half of the wide strip is opposite in direction to the shear $(\sigma_{r\theta})$ in the lower half of the wide strip, $(\sigma_{r\theta})$ is equal to zero when $(\theta = 0^\circ)$. [This applies for all ($\sigma_{r\theta}$): ($\sigma_{r\theta}^{II}$), ($\sigma_{r\theta}^{III}$) and ($\sigma_{r\theta}^{IV}$).]

$$\sigma_{r\theta}\Big|_{\theta=0^\circ} = 0 \tag{22}$$

Along the wide strip - die interface ($\theta = \alpha^\circ$), from ($r = r_f + \mathcal{E}_1$) to ($r = r_o - \mathcal{E}_2$), a constant shear stress (τ) is assumed. (Refer to Figure 3.)

$$\tau = m \frac{\sigma_o}{\sqrt{3}} \qquad 0 \leq m \leq 1 \tag{9}$$

This constant shear stress (τ) is equivalent to ($\sigma_{\theta r}$), and its direction is opposite to that of the flowing wide strip. (Since the shear stress (T_I) in Figure 2 is negative and is in the opposite direction of (τ), (τ) is a positive shear stress.) Therefore, for zone II at ($r = r_f + \mathcal{E}_1$) and ($\theta = \alpha^\circ$),

$$\tau = \sigma_{\theta r}^{II}\Big|_{\substack{r=r_f+\mathcal{E}_1 \\ \theta=\alpha^\circ}} = +m\frac{\sigma_o}{\sqrt{3}}. \tag{23}$$

Another boundary condition which ($\sigma_{r\theta}^{II}$) must meet is in the first equation of equations (18). At ($r = r_f$),

$$\sigma_{r\theta}^{II}\Big|_{r=r_f} = -\frac{1}{2}\sigma_{xx_I}\sin 2\theta. \tag{24}$$

From equation (24), at ($\theta = \alpha^\circ$),

$$\sigma_{\theta r}^{II}\Big|_{\substack{r=r_f \\ \theta=\alpha^\circ}} = -\frac{1}{2}\sigma_{xx_I}\sin 2\alpha. \tag{25}$$

Therefore, from equations (25) and (23), ($\sigma_{\theta r}^{II}/_{\theta=\alpha^\circ}$) changes from ($-\frac{1}{2}\sigma_{xx_I}\sin 2\alpha$) at ($r = r_f$) to ($+m\frac{\sigma_o}{\sqrt{3}}$) at ($r = r_f + \mathcal{E}_1$). This change is assumed to be linear with (r) as shown in Figure 3. Also, ($\sigma_{r\theta}^{II}$) as a function of (θ), at constant (r), is assumed throughout zone II. Since $(\sigma^{II}_{r\theta})$ varies with $(\sin 2\theta)$ at $(r = r_f)$, it is assumed that $(\sigma^{II}_{r\theta})$ varies with $(\sin 2\theta)$, at constant (r), in all of zone II.

A shear stress which meets the assumed distribution and the boundary conditions is

$$\sigma^{II}_{r\theta} = +\frac{1}{2}\sigma_{xx_I}\left[\frac{r-(r_f+\varepsilon_1)}{\varepsilon_1}\right](\sin 2\theta) + \frac{1}{\sqrt{3}}\left(\frac{m}{\sin 2\alpha}\right)\sigma_o\left[\frac{r-r_f}{\varepsilon_1}\right](\sin 2\theta) . \tag{26}$$

(σ^{II}_{rr}) and $(\sigma^{II}_{\theta\theta})$ can now be found from the equations of equilibrium [equations (21)] .

The derivative of $(\sigma^{II}_{r\theta})$ with respect to (r) is

$$\frac{\partial\sigma^{II}_{r\theta}}{\partial r} = \left[+\frac{1}{2}\sigma_{xx_I} + \left(\frac{m}{\sin 2\alpha}\right)\frac{\sigma_o}{\sqrt{3}}\right]\left(\frac{1}{\varepsilon_1}\right)(\sin 2\theta) \tag{27}$$

By substituting equations (26) and (27) into the second of equations (21), the partial derivative $(\partial\sigma^{II}_{\theta\theta}/\partial\theta)$ is found.

$$\frac{\partial\sigma^{II}_{\theta\theta}}{\partial\theta} = \left\{\left[+\frac{1}{2}\sigma_{xx_I} + \left(\frac{m}{\sin 2\alpha}\right)\frac{\sigma_o}{\sqrt{3}}\right]\left[\frac{2r_f-3r}{\varepsilon_1}\right] + \sigma_{xx_I}\right\}\sin 2\theta \tag{28}$$

Integrating equation (28) gives

$$\sigma^{II}_{\theta\theta} = \left\{\left[+\frac{1}{2}\sigma_{xx_I} + \left(\frac{m}{\sin 2\alpha}\right)\frac{\sigma_o}{\sqrt{3}}\right]\left[\frac{\frac{3}{2}r-r_f}{\varepsilon_1}\right] - \frac{1}{2}\sigma_{xx_I}\right\}\cos 2\theta + f^{II}(r) \tag{29}$$

where $(f^{II}(r))$ is a constant of integration. Since there are no known boundary conditions involving $(\sigma^{II}_{\theta\theta})$, $(f^{II}(r))$ cannot be determined yet. Therefore, the next step is to solve the first of equations (21) for (σ_{rr}^{II}).

Taking the terms $((\partial\sigma_{rr}/\partial r)+(\sigma_{rr}/r))$ out of the first of equations (21) and rearranging these terms makes the solution of this differential equation simpler. By inspection,

$$\left[\frac{\partial\sigma_{rr}}{\partial r}+\frac{\sigma_{rr}}{r}\right]=\frac{1}{r}\cdot\frac{\partial}{\partial r}\left[r\cdot\sigma_{rr}\right]. \tag{30}$$

Substituting equation (30) into the first of equations (21) leads to

$$\frac{\partial}{\partial r}\left[r\cdot\sigma_{rr}\right]=\sigma_{\theta\theta}-\frac{\partial\sigma_{r\theta}}{\partial\theta}. \tag{31}$$

To find (σ_{rr}^{II}), equation (31) is used in place of the first of equations (21).

The partial derivative of $(\sigma_{r\theta}^{II})$ (equation (26)) with respect to (θ) is

$$\frac{\partial\sigma_{r\theta}^{II}}{\partial\theta}=2\left\{\left[+\frac{1}{2}\sigma_{xx_I}+\left(\frac{m}{\sin 2\alpha}\right)\frac{\sigma_0}{\sqrt{3}}\right]\left[\frac{r-r_f}{\varepsilon_1}\right]-\frac{1}{2}\sigma_{xx_f}\right\}\cos 2\theta. \tag{32}$$

Substituting equations (29) and (32) into equation (31) gives

$$\frac{\partial}{\partial r}\left[r\cdot\sigma_{rr}^{II}\right]=$$

$$\left\{\left[+\frac{1}{2}\sigma_{xx_I}+\left(\frac{m}{\sin 2\alpha}\right)\frac{\sigma_0}{\sqrt{3}}\right]\left[\frac{r_f-\frac{1}{2}r}{\varepsilon_1}\right]+\frac{1}{2}\sigma_{xx_I}\right\}\cos 2\theta+f^{II}(r) \tag{33}$$

Integrating equation (33) and then dividing through by (r) yield (σ_{rr}^{II}).

$$\sigma_{rr}^{II}=\left\{\left[+\frac{1}{2}\sigma_{xx_I}+\left(\frac{m}{\sin 2\alpha}\right)\frac{\sigma_0}{\sqrt{3}}\right]\left[\frac{r_f-\frac{1}{4}r}{\varepsilon_1}\right]+\frac{1}{2}\sigma_{xx_I}\right\}\cos 2\theta+\left(\frac{1}{r}\right)\int f^{II}(r)\,dr+\left(\frac{1}{r}\right)f^{II}(\theta), \tag{34}$$

where $f^{II}(\Theta)$ is an integration constant.

In order to solve for the integration constant $[f^{II}(r)]$ in equations (29) and (34), the following substitution is made. Let

$$\left(\frac{1}{r}\right)\int f^{II}(r)\,dr = F^{II}(r)$$

$$\int f^{II}(r)\,dr = r\,F^{II}(r) \tag{35}$$

$$f^{II}(r) = r\,F'^{II}(r) + F^{II}(r)\,.$$

Therefore, from equation (29), $(\sigma^{II}_{\theta\theta})$ becomes

$$\sigma^{II}_{\theta\theta} = \left\{\left[+\frac{1}{2}\sigma_{xx_I} + \left(\frac{m}{\sin 2\alpha}\right)\frac{\sigma_0}{\sqrt{3}}\right]\left[\frac{3/2\,r - r_f}{\varepsilon_1}\right] - \frac{1}{2}\sigma_{xx_I}\right\}\cos 2\theta + r\,F'^{II}(r) + F^{II}(r)\,, \tag{36}$$

and from equation (34), (σ^{II}_{rr}) becomes

$$\sigma^{II}_{rr} = \left\{\left[+\frac{1}{2}\sigma_{xx_I} + \left(\frac{m}{\sin 2\alpha}\right)\frac{\sigma_0}{\sqrt{3}}\right]\left[\frac{r_f - \frac{1}{4}r}{\varepsilon_1}\right] + \frac{1}{2}\sigma_{xx_I}\right\}\cos 2\theta + F^{II}(r) + \left(\frac{1}{r}\right)f^{II}(\theta). \tag{37}$$

The unknown functions in equations (36) and (37) are found by satisfying the von Mises yield criterion in cylindrical coordinates with plane strain conditions and the assumption of yielding at $(\theta = 0^\circ)$. (Assuming yielding to occur at $(\theta = 0^\circ)$ instead of $(\theta = \alpha^\circ)$ keeps the solution much simpler.)

The von Mises yield criterion for a cylindrical coordinate system is

$$\frac{1}{6}\left[(\sigma_{rr} - \sigma_{\theta\theta})^2 + (\sigma_{\theta\theta} - \sigma_{zz})^2 + (\sigma_{zz} - \sigma_{rr})^2\right] + \sigma_{r\theta}^2 + \sigma_{\theta z}^2 + \sigma_{zr}^2 \leq \frac{\sigma_0^2}{3}\,. \tag{3}$$

In plane strain, the yield criterion (equation (3)) becomes

$$-\sigma_{rr} - \sigma_{\theta\theta} \leq 2\sqrt{\tfrac{1}{3}\sigma_0^2 - \sigma_{r\theta}^2}\,. \tag{38}$$

(Equation (38) is derived in Appendix I.)

Evaluating ($\sigma_{\theta\theta}^{II}$) from equation (36), ($\sigma_{rr}^{II}$) from equation (37) and ($\sigma_{r\theta}^{II}$) from equation (26) at ($\theta = 0°$), and then substituting them into equation (38) gives

$$\left[+\tfrac{1}{2}\sigma_{xx_I} + \left(\frac{m}{\sin 2\alpha}\right)\frac{\sigma_0}{\sqrt{3}}\right]\left[\frac{2r_f - 7/4\, r}{\varepsilon_1}\right] + \sigma_{xx_I} - r\cdot F'^{II}(r) + \left(\frac{1}{r}\right) f^{II}(0°) = \frac{2}{\sqrt{3}}\sigma_0 = 2K. \tag{39}$$

The term ($r\cdot F'^{II}(r)$) is determined from equation (39).

$$r\cdot F'^{II}(r) = \left[+\tfrac{1}{2}\sigma_{xx_I} + \left(\frac{m}{\sin 2\alpha}\right)\frac{\sigma_0}{\sqrt{3}}\right]\left[\frac{2r_f - 7/4\, r}{\varepsilon_1}\right] + \sigma_{xx_I} + \left(\frac{1}{r}\right) f^{II}(0°) - \frac{2}{\sqrt{3}}\sigma_0 \tag{40}$$

Dividing through by (r) and integrating yields $F^{II}(r)$.

$$F^{II}(r) = -\frac{7}{4}\left[+\tfrac{1}{2}\sigma_{xx_I} + \left(\frac{m}{\sin 2\alpha}\right)\frac{\sigma_0}{\sqrt{3}}\right]\left(\frac{r}{\varepsilon_1}\right) - \left(\frac{1}{r}\right) f^{II}(0°) + \left\{2\left[+\tfrac{1}{2}\sigma_{xx_I} + \left(\frac{m}{\sin 2\alpha}\right)\frac{\sigma_0}{\sqrt{3}}\right]\left(\frac{r_f}{\varepsilon_1}\right) + \sigma_{xx_I} - \frac{2}{\sqrt{3}}\sigma_0\right\}\ln r + C_I \tag{41}$$

where (C_I) is an integration constant. To prevent unit difficulties, (C_I) is assumed to be

$$C_I = \left\{2\left[+\tfrac{1}{2}\sigma_{xx_I} + \left(\frac{m}{\sin 2\alpha}\right)\frac{\sigma_0}{\sqrt{3}}\right]\left(\frac{r_f}{\varepsilon_1}\right) + \sigma_{xx_I} - \frac{2}{\sqrt{3}}\sigma_0\right\}\ln\left(\frac{1}{c r_f}\right) \tag{42}$$

where c is a constant to be found later. Combining equations (41) and (42) gives

$$F^{II}(r) = -\frac{7}{4}\left[+\frac{1}{2}\sigma_{xx_I} + \left(\frac{m}{\sin 2\alpha}\right)\frac{\sigma_o}{\sqrt{3}}\right]\left(\frac{r}{\varepsilon_1}\right) - \left(\frac{1}{r}\right) f^{II}(0^\circ)$$

$$+\left\{2\left[+\frac{1}{2}\sigma_{xx_I} + \left(\frac{m}{\sin 2\alpha}\right)\frac{\sigma_o}{\sqrt{3}}\right]\left(\frac{r_f}{\varepsilon_1}\right) + \sigma_{xx_I} - \frac{2}{\sqrt{3}}\sigma_o\right\}\ln\left(\frac{r}{c r_f}\right). \tag{43}$$

By substituting equation (43) into equation (37), (σ_{rr}^{II}) is rewritten as

$$\sigma_{rr}^{II} = \left\{\left[+\frac{1}{2}\sigma_{xx_I} + \left(\frac{m}{\sin 2\alpha}\right)\frac{\sigma_o}{\sqrt{3}}\right]\left[\frac{r_f - \frac{1}{4}r}{\varepsilon_1}\right] + \frac{1}{2}\sigma_{xx_I}\right\}\cos 2\theta$$

$$-\frac{7}{4}\left[+\frac{1}{2}\sigma_{xx_I} + \left(\frac{m}{\sin 2\alpha}\right)\frac{\sigma_o}{\sqrt{3}}\right]\left(\frac{r}{\varepsilon_1}\right) + \left(\frac{1}{r}\right)\left[f^{II}(\theta) - f^{II}(0^\circ)\right] \tag{44}$$

$$+\left\{2\left[+\frac{1}{2}\sigma_{xx_I} + \left(\frac{m}{\sin 2\alpha}\right)\frac{\sigma_o}{\sqrt{3}}\right]\left(\frac{r_f}{\varepsilon_1}\right) + \sigma_{xx_I} - \frac{2}{\sqrt{3}}\sigma_o\right\}\ln\left(\frac{r}{c r_f}\right).$$

The function $[f^{II}(\theta) - f^{II}(0^\circ)]$ and the constant $\ln(1/c)$ can now be determined.

From the second equation of equations (18), at $(r = r_f)$,

$$\sigma_{rr}^{II}\Big|_{r=r_f} = +\sigma_{xx_I}\cos^2\theta \tag{45}$$

Inserting this boundary condition into equation (44) allows the function $[f^{II}(\theta) - f^{II}(0^\circ)]$ to be found.

$$\left[f^{II}(\theta) - f^{II}(0^\circ)\right] = +\frac{1}{2}\sigma_{xx_I}(r_f) + \frac{7}{4}\left[+\frac{1}{2}\sigma_{xx_I} + \left(\frac{m}{\sin 2\alpha}\right)\frac{\sigma_o}{\sqrt{3}}\right]\left(\frac{r_f}{\varepsilon_1}\right) r_f$$

$$-\frac{3}{4}\left[+\frac{1}{2}\sigma_{xx_I} + \left(\frac{m}{\sin 2\alpha}\right)\frac{\sigma_o}{\sqrt{3}}\right]\left(\frac{r_f}{\varepsilon_1}\right) r_f \cos 2\theta \tag{46}$$

$$-\left\{2\left[+\frac{1}{2}\sigma_{xx_I} + \left(\frac{m}{\sin 2\alpha}\right)\frac{\sigma_o}{\sqrt{3}}\right]\left(\frac{r_f}{\varepsilon_1}\right) + \sigma_{xx_I} - \frac{2}{\sqrt{3}}\sigma_o\right\}(r_f)\ln\left(\frac{1}{c}\right)$$

At $(\theta = 0^\circ)$, $[f^{II}(\theta) - f^{II}(0^\circ)] = 0$, and the constant $\ln(\frac{1}{c})$ is determined from equation (46).

$$\ln\left(\frac{1}{c}\right) = \frac{\left[+\frac{1}{2}\sigma_{xx_I} + \left(\frac{m}{\sin 2\alpha}\right)\frac{\sigma_o}{\sqrt{3}}\right]\left(\frac{r_f}{\varepsilon_1}\right) + \frac{1}{2}\sigma_{xx_I}}{2\left[+\frac{1}{2}\sigma_{xx_I} + \left(\frac{m}{\sin 2\alpha}\right)\frac{\sigma_o}{\sqrt{3}}\right]\left(\frac{r_f}{\varepsilon_1}\right) + \sigma_{xx_I} - \frac{2\sigma_o}{\sqrt{3}}} \tag{47}$$

The stress ratios $(\sigma_{rr}^{II}/\sigma_o)$ and $(\sigma_{\theta\theta}^{II}/\sigma_o)$ are now obtainable.

Substituting equations (46) and (47) into equation (44) and then dividing both sides of the resulting equation by the yield stress in uniaxial tension (σ_o) results in

$$
\begin{aligned}
\frac{\sigma_{rr}^{II}}{\sigma_o} = {} & \left[+\left(\frac{\sigma_{xx_I}}{\sigma_o}\right)+\frac{2}{\sqrt{3}}\left(\frac{m}{\sin 2\alpha}\right)\right]\left[\frac{3+4\left(\frac{r}{r_f}\right)-7\left(\frac{r}{r_f}\right)^2}{8\left(\frac{r}{r_f}\right)\left(\frac{\varepsilon_1}{r_f}\right)}\right] \\
& + \left\{\left[+\left(\frac{\sigma_{xx_I}}{\sigma_o}\right)+\frac{2}{\sqrt{3}}\left(\frac{m}{\sin 2\alpha}\right)\right]\left[\frac{1}{(\varepsilon_1/r_f)}\right]+\left(\frac{\sigma_{xx_I}}{\sigma_o}\right)-\frac{2}{\sqrt{3}}\right\}\ln\left(\frac{r}{r_f}\right) \qquad (48) \\
& + \left[+\left(\frac{\sigma_{xx_I}}{\sigma_o}\right)+\frac{2}{\sqrt{3}}\left(\frac{m}{\sin 2\alpha}\right)\right]\left[\frac{-3+4\left(\frac{r}{r_f}\right)-\left(\frac{r}{r_f}\right)^2}{8\left(\frac{r}{r_f}\right)\left(\frac{\varepsilon_1}{r_f}\right)}\right]\cos 2\theta \\
& + \frac{1}{2}\left(\frac{\sigma_{xx_I}}{\sigma_o}\right)(1+\cos 2\theta).
\end{aligned}
$$

The stress $(\sigma_{\theta\theta}^{II})$ is found by substituting equations (40), (43) and (47) into equation (36). This equation divided by (σ_o) gives

$$
\begin{aligned}
\frac{\sigma_{\theta\theta}^{II}}{\sigma_o} = {} & \left[+\left(\frac{\sigma_{xx_I}}{\sigma_o}\right)+\frac{2}{\sqrt{3}}\left(\frac{m}{\sin 2\alpha}\right)\right]\left[\frac{6-7\left(\frac{r}{r_f}\right)}{4(\varepsilon_1/r_f)}\right] + \left\{\left[+\left(\frac{\sigma_{xx_I}}{\sigma_o}\right)+\frac{2}{\sqrt{3}}\left(\frac{m}{\sin 2\alpha}\right)\right]\cdot \right. \\
& \left.\left[\frac{1}{(\varepsilon_1/r_f)}\right]+\left(\frac{\sigma_{xx_I}}{\sigma_o}\right)-\frac{2}{\sqrt{3}}\right\}\ln\left(\frac{r}{r_f}\right)+\left[+\left(\frac{\sigma_{xx_I}}{\sigma_o}\right)+\frac{2}{\sqrt{3}}\left(\frac{m}{\sin 2\alpha}\right)\right]\cdot \qquad (49) \\
& \left[\frac{-2+3(r/r_f)}{4(\varepsilon_1/r_f)}\right]\cos 2\theta - \frac{1}{2}\left(\frac{\sigma_{xx_I}}{\sigma_o}\right)\cos 2\theta \\
& + 3/2\left(\frac{\sigma_{xx_I}}{\sigma_o}\right) - 2/\sqrt{3}
\end{aligned}
$$

Equations ((h) in Appendix I), (48) and (49) are used to determine the stress ratio $(\sigma_{zz}^{II}/\sigma_o)$ if it is desired.

If the stress field in zone II is to be statically admissible, it must satisfy the conditions on page 8. Requirements (1), (2) and (3) are obeyed by the stresses (σ_{rr}^{II}), $(\sigma_{\theta\theta}^{II})$ and $(\sigma_{r\theta}^{II})$. However, the fact that the von Mises yield criterion must not only be satisfied at

$(\theta = 0°)$ but throughout zone II is used later in the determination of the maximum allowable drawing stress. Therefore, the derivation of the drawing stress is continued by solving for the stress field in zone III.

In order to determine the stress field in zone III, again the distribution of the shear stress $(\sigma^{III}_{r\theta})$ is assumed. (Refer to Figure 3.) From $(r = r_f + \varepsilon_1)$ to $(r = r_o - \varepsilon_2)$ at $(\theta = \alpha°)$, constant friction is assumed, as stated earlier. In other words, $(\sigma^{III}_{\theta r})$ is constant along the die-wide strip interface, and it is equal to

$$\sigma^{III}_{\theta r}\Big|_{\theta=\alpha°} = \tau = + m \frac{\sigma_o}{\sqrt{3}} . \tag{50}$$

Since the assumed shear stress in zone II $(\sigma^{II}_{r\theta})$ varies with $(\sin 2\theta)$ at constant (r), it is assumed that $(\sigma^{III}_{r\theta})$ also varies with $(\sin 2\theta)$, at constant (r), throughout zone III. Therefore, the shear stress distribution in zone III is

$$\sigma^{III}_{r\theta} = + \frac{1}{\sqrt{3}}\left(\frac{m}{\sin 2\alpha}\right)\sigma_o\,(\sin 2\theta) . \tag{51}$$

By use of the shear variation of $(\sigma^{III}_{r\theta})$, the equilibrium equations [equations (21)] are again solved to determine the stresses (σ^{III}_{rr}) and $(\sigma^{III}_{\theta\theta})$. The procedure is exactly the same as that used for zone II, and therefore the explanation is somewhat more brief.

The derivative of $(\sigma^{III}_{r\theta})$ with respect to (r) is

$$\frac{\partial \sigma^{III}_{r\theta}}{\partial r} = 0 . \tag{52}$$

Substituting equations (51) and (52) into the second of the equations of equilibrium [equations (21)] gives

$$\frac{\partial \sigma_{\theta\theta}^{III}}{\partial \theta} = -2\left(\frac{m}{\sin 2\alpha}\right)\frac{\sigma_o}{\sqrt{3}}\left(\sin 2\theta\right). \tag{53}$$

Upon integration, $(\sigma_{\theta\theta}^{III})$ is found.

$$\sigma_{\theta\theta}^{III} = +\left(\frac{m}{\sin 2\alpha}\right)\frac{\sigma_o}{\sqrt{3}}\left(\cos 2\theta\right) + f^{III}(r) \tag{54}$$

where $f^{III}(r)$ is an integration constant. Since $f^{III}(r)$ cannot be determined at this point, (σ_{rr}^{III}) will be found from the first of the equilibrium equations [equations (21)].

The derivative of $(\sigma_{r\theta}^{III})$ with respect to (θ) is

$$\frac{\partial \sigma_{r\theta}^{III}}{\partial \theta} = +2\left(\frac{m}{\sin 2\alpha}\right)\frac{\sigma_o}{\sqrt{3}}\left(\cos 2\theta\right). \tag{55}$$

The substitution of equations (54) and (55) into equation (31) [(Equation(31)]is the first of the equilibrium equations [equations (21)] after rearrangement.) gives

$$\frac{\partial}{\partial r}\left[r \cdot \sigma_{rr}^{III}\right] = -\left(\frac{m}{\sin 2\alpha}\right)\frac{\sigma_o}{\sqrt{3}}\left(\cos 2\theta\right) + f^{III}(r). \tag{56}$$

The solution of (σ_{rr}^{III}) is found by integrating equation (56) and then dividing through by (r).

$$\sigma_{rr}^{III} = -\left(\frac{m}{\sin 2\alpha}\right)\frac{\sigma_o}{\sqrt{3}}\left(\cos 2\theta\right) + \left(\frac{1}{r}\right)\int f^{III}(r)\,dr + \left(\frac{1}{r}\right) f^{III}(\theta) \tag{57}$$

where $[f^{III}(\theta)]$ is a constant of integration. Since

$$\left(\frac{1}{r}\right)\int f^{III}(r)\,dr = F^{III}(r) \quad \text{and}$$

$$f^{III}(r) = r \cdot F'^{III}_{(r)} + F^{III}_{(r)} \tag{58}$$

[from equations (35)], (σ^{III}_{rr}) becomes

$$\sigma^{III}_{rr} = -\left(\frac{m}{\sin 2\alpha}\right)\frac{\sigma_o}{\sqrt{3}}(\cos 2\theta) + F^{III}_{(r)} + \left(\frac{1}{r}\right) f^{III}(\theta) \tag{59}$$

To find $(rF'^{III}_{(r)})$, $(F^{III}_{(r)})$ and $[f^{III}(\theta) - f^{III}(0°)]$, the von Mises yield criterion, with the assumption of yielding at $(\theta = 0°)$, is satisfied.

At $(\theta = 0°)$, (σ^{III}_{rr}), and $(\sigma^{III}_{\theta\theta})$ become

$$\sigma^{III}_{rr}\Big|_{\theta=0°} = -\left(\frac{m}{\sin 2\alpha}\right)\frac{\sigma_o}{\sqrt{3}} + F^{III}_{(r)} + \left(\frac{1}{r}\right) f^{III}_{(0°)} \tag{60}$$

$$\sigma^{III}_{\theta\theta}\Big|_{\theta=0°} = +\left(\frac{m}{\sin 2\alpha}\right)\frac{\sigma_o}{\sqrt{3}} + f^{III}(r)\,. \tag{61}$$

However, since $f^{III}(r) = r\,F'^{III}(r) + F^{III}(r)$, (58)

$$\sigma^{III}_{\theta\theta}\Big|_{\theta=0°} = +\left(\frac{m}{\sin 2\alpha}\right)\frac{\sigma_o}{\sqrt{3}} + r \cdot F'^{III}_{(r)} + F^{III}_{(r)} \tag{62}$$

Also, at $(\theta = 0°)$, from equation (51),

$$\left(\sigma^{III}_{r\theta}\right)^2\Big|_{\theta=0°} = 0. \tag{63}$$

Therefore, with the assumption of yielding at $(\theta = 0°)$, the von Mises yield criterion

$$\sigma^{III}_{rr} - \sigma^{III}_{\theta\theta} \leq 2\sqrt{\frac{1}{3}\sigma_o^2 - \left(\sigma^{III}_{r\theta}\right)^2} \tag{38}$$

becomes

$$-2\left(\frac{m}{\sin 2\alpha}\right)\frac{\sigma_0}{\sqrt{3}} + \left(\frac{1}{r}\right) f^{III}(0^\circ) - r \cdot F'^{III}(r) = \frac{2}{\sqrt{3}}\sigma_0 . \tag{64}$$

Solving for $[rF'^{III}(r)]$ gives

$$r \cdot F'^{III}(r) = -2\left(\frac{m}{\sin 2\alpha}\right)\frac{\sigma_0}{\sqrt{3}} + \left(\frac{1}{r}\right) f^{III}(0^\circ) - \frac{2}{\sqrt{3}}\sigma_0 . \tag{65}$$

Dividing equation (65) by (r) and then integrating yields

$$F^{III}(r) = -\left(\frac{1}{r}\right) f^{III}(0^\circ) - \left[2\left(\frac{m}{\sin 2\alpha}\right)\frac{\sigma_0}{\sqrt{3}} + \frac{2}{\sqrt{3}}\sigma_0\right] \ln r + C_{II} \tag{66}$$

where (C_{II}) is an integration constant.

To keep the solution dimensionless, let

$$C_{II} = -\left[2\left(\frac{m}{\sin 2\alpha}\right)\frac{\sigma_0}{\sqrt{3}} + \frac{2}{\sqrt{3}}\sigma_0\right] \ln\left(\frac{1}{a r_f}\right) . \tag{67}$$

where (a) is also a constant.

Therefore,

$$F^{III}(r) = -\left(\frac{1}{r}\right) f^{III}(0^\circ) - \left[2\left(\frac{m}{\sin 2\alpha}\right)\frac{\sigma_0}{\sqrt{3}} + \frac{2}{\sqrt{3}}\sigma_0\right] \ln\left(\frac{r}{a r_f}\right) . \tag{68}$$

Substituting $[F^{III}(r)]$ from equation (68) into equation (59) allows (σ_{rr}^{III}) to be found as a function of the unknown constant (a) and the function $\left[f^{III}(\theta) - f^{III}(0^\circ)\right]$.

$$\sigma_{rr}^{III} = -\left(\frac{m}{\sin 2\alpha}\right)\frac{\sigma_0}{\sqrt{3}}(\cos 2\theta) - \left[2\left(\frac{m}{\sin 2\alpha}\right)\frac{\sigma_0}{\sqrt{3}} + \frac{2}{\sqrt{3}}\sigma_0\right] \ln\left(\frac{r}{a r_f}\right) + \left(\frac{1}{r}\right)\left[f^{III}(\theta) - f^{III}(0^\circ)\right] \tag{69}$$

Since the stress field in zone II imposes a boundary condition on zone III at $(r = r_f + \mathcal{E}_1)$, (σ_{rr}^{III}) must meet this requirement. At $(r = r_f + \mathcal{E}_1)$,

$$\sigma_{rr}^{III}\Big|_{r_f+\varepsilon_1} = \sigma_{rr}^{II}\Big|_{r_f+\varepsilon_1} . \tag{70}$$

Therefore, at $(r = r_f + \mathcal{E}_1)$, equation (69) is used to determine $[f^{III}(\Theta) - f^{III}(0°)]$.

$$\left[f^{III}(\Theta) - f^{III}(0°)\right] = (r_f+\varepsilon_1)\left\{\sigma_{rr}^{II}\Big|_{r_f+\varepsilon_1} + \left(\frac{m}{\sin 2\alpha}\right)\frac{\sigma_0}{\sqrt{3}}(\cos 2\Theta) + \left[2\left(\frac{m}{\sin 2\alpha}\right)\frac{\sigma_0}{\sqrt{3}} + \frac{2\sigma_0}{\sqrt{3}}\right]\ln\left[1+\left(\frac{\varepsilon_1}{r_f}\right)\right]\right\} + (r_f+\varepsilon_1)\left[2\left(\frac{m}{\sin 2\alpha}\right)\frac{\sigma_0}{\sqrt{3}} + \frac{2}{\sqrt{3}}\sigma_0\right]\ln\left(\frac{1}{a}\right) \tag{71}$$

At $(\Theta = 0°)$, $[f^{III}(\Theta) - f^{III}(0°)] = 0$, and the constant term $\ln(\frac{1}{a})$ is found to be

$$\ln\left(\frac{1}{a}\right) = -\frac{\left\{\sigma_{rr}^{II}\Big/_{r_f+\varepsilon_1}^{\Theta=0°} + \left(\frac{m}{\sin 2\alpha}\right)\frac{\sigma_0}{\sqrt{3}} + \left[2\left(\frac{m}{\sin 2\alpha}\right)\frac{\sigma_0}{\sqrt{3}} + \frac{2}{\sqrt{3}}\sigma_0\right]\ln\left[1+\left(\frac{\varepsilon_1}{r_f}\right)\right]\right\}}{2\left(\frac{m}{\sin 2\alpha}\right)\frac{\sigma_0}{\sqrt{3}} + \frac{2}{\sqrt{3}}\sigma_0} . \tag{72}$$

(σ_{rr}^{III}) is determined by substituting equations (71) and (72) into equation (69). However, (σ_{rr}^{III}) is still in terms of $(\sigma_{rr}^{II}/_{r_f+\varepsilon_1}^{\Theta=0°})$ and $(\sigma_{rr}^{II}/_{r_f+\varepsilon_1})$. After these evaluations of equation (48) are made, and (σ_{rr}^{III}) is divided by (σ_0), the stress ratio $(\sigma_{rr}^{III}/\sigma_0)$ is

$$\frac{\sigma_{rr}^{III}}{\sigma_0} = -\left[\left(\frac{\sigma_{xx_I}}{\sigma_0}\right) + \frac{2}{\sqrt{3}}\left(\frac{m}{\sin 2\alpha}\right)\right]\left[\frac{6+3\left(\frac{\varepsilon_1}{r_f}\right)}{8\left(\frac{r}{r_f}\right)}\right](1-\cos 2\Theta) - \frac{2}{\sqrt{3}}\left[1+\left(\frac{m}{\sin 2\alpha}\right)\right]\ln\left(\frac{r}{r_f}\right) - \frac{1}{\sqrt{3}}\left(\frac{m}{\sin 2\alpha}\right)(1+\cos 2\Theta) \tag{73}$$

$$+\left[\left(\frac{\sigma_{xx_I}}{\sigma_o}\right)+\frac{2}{\sqrt{3}}\left(\frac{m}{\sin 2\alpha}\right)\right]\left[\frac{1+\left(\frac{\mathcal{E}_1}{r_f}\right)}{\left(\mathcal{E}_1/r_f\right)}\right]\ln\left[1+\left(\frac{\mathcal{E}_1}{r_f}\right)\right].$$

$\left(\frac{\sigma_{\theta\theta}^{III}}{\sigma_o}\right)$ is found by evaluating equation (48) at $(r = r_f + \mathcal{E}_1)$, $(\theta = 0°)$ and then substituting equations (58), (68), (65) and (72) into equation (54), after it has been divided by (σ_o).

$$\frac{\sigma_{\theta\theta}^{III}}{\sigma_o} = +\frac{2}{\sqrt{3}}\left[1+\left(\frac{m}{\sin 2\alpha}\right)\right]\ln\left[\frac{1+(\mathcal{E}_1/r_f)}{\left(\frac{r}{r_f}\right)}\right]+\frac{1}{\sqrt{3}}\left(\frac{m}{\sin 2\alpha}\right)\left(-3+\cos 2\theta\right)$$
$$-\frac{2}{\sqrt{3}}+\left\{\left(\frac{\sigma_{xx_I}}{\sigma_o}\right)\left[1+\frac{1}{\left(\frac{\mathcal{E}_1}{r_f}\right)}\right]+\frac{2}{\sqrt{3}}\left[\frac{\left(\frac{m}{\sin 2\alpha}\right)}{\left(\frac{\mathcal{E}_1}{r_f}\right)}-1\right]\right\}\ln\left[1+\left(\frac{\mathcal{E}_1}{r_f}\right)\right] \qquad (74)$$

$\left(\frac{\sigma_{zz}^{III}}{\sigma_o}\right)$ can be found from equation (h) in Appendix I and equations (73) and (74).

As in zone II, the stress field in zone III must meet the requirements for a statically admissible stress field. The stresses (σ_{rr}^{III}), $(\sigma_{\theta\theta}^{III})$ and $(\sigma_{r\theta}^{III})$ meet conditions (1), (2) and (3) on page 8 as well as satisfying the von Mises yield criterion at $(\theta = 0°)$. However, the attempt to satisfy the von Mises yield criterion throughout zone III is not considered until all stress fields have been determined.

Before the stress field in zone IV can be found, the boundary conditions at $(r = r_o)$ must be known. They are determined by resolving the back tension stress (σ_{xxV}) into its normal and tangential components, (N_V) and (T_V), respectively. (Refer to Figure 2.) The procedure for determining (N_V) and (T_V) is exactly the same as that used in zone I for finding (N_I) and (T_I).

The nine components of stress is zone V are: $\sigma_{11} = \sigma_{xxV}$;

$\sigma_{22} = \sigma_{12} = \sigma_{23} = \sigma_{31} = 0.$

$$\sigma_{33} = \frac{1}{2}\left(\sigma_{11} + \sigma_{22}\right) = \frac{1}{2}\sigma_{11} = \frac{1}{2}\,\sigma_{xx_{V}} \tag{75}$$

Using equations (11), (12), (14) and (15), the normal and shear stresses in zone V are found to be

$$\begin{aligned} N_{V} &= +\,\sigma_{xx_{V}}\cos^2\theta \\ T_{V} &= -\frac{1}{2}\sigma_{xx_{V}}(\sin 2\theta)\,. \end{aligned} \tag{76}$$

Therefore, at ($r = r_o$), or at the surface of stress discontinuity (s_4),

$$\begin{aligned} N_{V} &= \sigma_{rr}^{IV}\Big|_{r=r_o} = +\,\sigma_{xx_{V}}\cos^2\theta \\ T_{V} &= \sigma_{r\theta}^{IV}\Big|_{r=r_o} = -\frac{1}{2}\sigma_{xx_{V}}(\sin 2\theta)\,. \end{aligned} \tag{77}$$

The stress field in zone IV must satisfy the boundary conditions at ($r = r_o$) and ($r = r_o - \varepsilon_2$). At ($r = r_o$), equations (77) must be obeyed, and at ($r = r_o - \varepsilon_2$),

$$\sigma_{rr}^{III}\Big|_{r=r_o-\varepsilon_2} = \sigma_{rr}^{IV}\Big|_{r=r_o-\varepsilon_2}\,. \tag{78}$$

Therefore, the distribution of shear for ($\sigma_{r\theta}^{IV}$) is assumed, as was done for ($\sigma_{r\theta}^{II}$) and ($\sigma_{r\theta}^{III}$). However, since (σ_{rr}^{IV}) must obey two boundary conditions [(σ_{rr}^{II}) and (σ_{rr}^{III}) had to satisfy only one boundary condition], it is much easier to determine the remaining part of the stress field in zone IV by assuming (σ_{rr}^{IV}) instead of solving for it. By assuming ($\sigma_{r\theta}^{IV}$) and (σ_{rr}^{IV}), the two equilibrium equations [equations (21)] have only one unknown stress remaining: ($\sigma_{\theta\theta}^{IV}$).

The boundary conditions which must be met by ($\sigma_{r\theta}^{IV}$) are

$$\sigma_{r\theta}^{IV}\Big|_{r=r_0} = -\frac{1}{2}\sigma_{xx_V}(\sin 2\theta) \tag{79}$$

$$\sigma_{\theta r}^{IV}\Big|_{\substack{r=r_0\\ \theta=\alpha^\circ}} = -\frac{1}{2}\sigma_{xx_V}(\sin 2\alpha) \tag{80}$$

$$\sigma_{\theta r}^{IV}\Big|_{\substack{r=r_0-\varepsilon_2\\ \theta=\alpha^\circ}} = +m\frac{\sigma_0}{\sqrt{3}}\ . \tag{81}$$

Also, as stated earlier, all ($\sigma_{r\theta}$) are equal to zero when ($\theta = 0^\circ$). The assumed distribution of ($\sigma_{r\theta}^{IV}$) is similar to that of ($\sigma_{r\theta}^{II}$). At constant (r), ($\sigma_{r\theta}^{IV}$) varies with ($\sin 2\theta$), and at ($\theta = \alpha^\circ$), ($\sigma_{\theta r}^{IV}$) is linearly dependent on (r) from ($r = r_0 - \varepsilon_2$) to ($r = r_0$). (Refer to Figure 3.) Therefore, a shear stress which satisfies the boundary conditions and the assumed distribution is

$$\sigma_{r\theta}^{IV} = \frac{1}{2}\sigma_{xx_V}\left[\frac{(r_0-\varepsilon_2)-r}{\varepsilon_2}\right](\sin 2\theta) + \left(\frac{m}{\sin 2\alpha}\right)\frac{\sigma_0}{\sqrt{3}}\left[\frac{r_0-r}{\varepsilon_2}\right](\sin 2\theta). \tag{82}$$

The boundary conditions at ($r = r_0 - \varepsilon_2$) and ($r = r_0$) which must be satisfied by (σ_{rr}^{IV}) are

$$\sigma_{rr}^{IV}\Big|_{r=r_0-\varepsilon_2} = \sigma_{rr}^{III}\Big|_{r=r_0-\varepsilon_2} \tag{78}$$

$$\sigma_{rr}^{IV}\Big|_{r=r_0} = \sigma_{xx_V} \cos^2\theta \, . \tag{83}$$

The distribution of (σ_{rr}^{IV}) which is assumed is that (σ_{rr}^{IV}) is a linear function of (r) from ($r = r_0 - \varepsilon_2$) to ($r = r_0$). The normal stress (σ_{rr}^{IV}) satisfying the boundary conditions and the assumed distribution is

$$\sigma_{rr}^{IV} = \sigma_{rr}^{III}\Big|_{r_0-\varepsilon_2}\left[\frac{r_0-r}{\varepsilon_2}\right] - \sigma_{xx_V}\cos^2\theta\left[\frac{(r_0-\varepsilon_2)-r}{\varepsilon_2}\right]. \tag{84}$$

The equilibrium equations [equations (21)] are combined and then rearranged to form a first-order, linear differential equation for ($\sigma_{\theta\theta}^{IV}$).

$$\frac{\partial \sigma_{\theta\theta}^{IV}}{\partial\theta} + \sigma_{\theta\theta}^{IV} = Q(r,\theta) \tag{85}$$

where

$$Q(r,\theta) = \frac{\partial}{\partial r}\left[r\cdot\sigma_{rr}^{IV}\right] + \frac{\partial \sigma_{r\theta}^{IV}}{\partial\theta} - r\left(\frac{\partial \sigma_{r\theta}^{IV}}{\partial r}\right) - 2\,\sigma_{r\theta}^{IV}$$

The solution of equation (85) from Golomb and Shanks (11) is

$$\sigma_{\theta\theta}^{IV} = \frac{1}{e^{\theta}}\left[\int Q(r,\theta)\cdot e^{\theta}\,d\theta + f^{IV}(r)\right] \tag{86}$$

where $[f^{IV}(r)]$ is an integration constant. Therefore, to find $(\sigma_{\theta\theta}^{IV})$ from equation (86), the function $[Q(r,\theta)]$ must be re-written.

The partial derivative of $(r \cdot \sigma_{rr}^{IV})$ with respect to (r) is

$$\frac{\partial}{\partial r}\left[r \cdot \sigma_{rr}^{IV}\right] = \sigma_{rr}^{III}\Big/_{r_0-\varepsilon_2}\left[\frac{r_0-2r}{\varepsilon_2}\right] - \sigma_{xx_V}\cos^2\theta\left[\frac{(r_0-\varepsilon_2)-2r}{\varepsilon_2}\right]. \tag{87}$$

Differentiating $(\sigma_{r\theta}^{IV})$ with respect to (θ) and (r) gives the following two equations:

$$\frac{\partial\sigma_{r\theta}^{IV}}{\partial\theta} = \sigma_{xx_V}\left[\frac{(r_0-\varepsilon_2)-r}{\varepsilon_2}\right](\cos 2\theta) + 2\left(\frac{m}{\sin 2\alpha}\right)\frac{\sigma_0}{\sqrt{3}}\left[\frac{r_0-r}{\varepsilon_2}\right](\cos 2\theta) \tag{88}$$

$$\frac{\partial\sigma_{r\theta}^{IV}}{\partial r} = -\frac{1}{2}\sigma_{xx_V}\left(\frac{1}{\varepsilon_2}\right)(\sin 2\theta) - \left(\frac{m}{\sin 2\alpha}\right)\frac{\sigma_0}{\sqrt{3}}\left(\frac{1}{\varepsilon_2}\right)(\sin 2\theta) \tag{89}$$

Substituting equations (82), (87), (88) and (89) into equation (85) yields $[Q(r,\theta)]$.

$$Q(r,\theta) = A_s \sin 2\theta + A_c \cos 2\theta + A_K$$

where

$$A_s = \left[\sigma_{xx_V} + \frac{2}{\sqrt{3}}\sigma_0\left(\frac{m}{\sin 2\alpha}\right)\right]\left[\frac{3\left(\frac{r}{r_f}\right) - 2\left(\frac{r_0}{r_f}\right)}{2\left(\frac{\varepsilon_2}{r_f}\right)}\right] + \sigma_{xx_V}$$

$$A_c = \left[\sigma_{xx_I} + \frac{2}{\sqrt{3}}\sigma_0\left(\frac{m}{\sin 2\alpha}\right)\right]\left[\frac{\left\langle\left(\frac{r_0}{r_f}\right) - 2\left(\frac{r}{r_f}\right)\right\rangle\left\langle 6 + 3\left(\frac{\varepsilon_1}{r_f}\right)\right\rangle}{8\left(\frac{\varepsilon_2}{r_f}\right)\left\langle\left(\frac{r_0}{r_f}\right) - \left(\frac{\varepsilon_2}{r_f}\right)\right\rangle}\right] +$$

$$\left[\sigma_{xx_{V}} + \frac{2}{\sqrt{3}}\sigma_o\left(\frac{m}{\sin 2\alpha}\right)\right]\left[\frac{\left(\frac{r_o}{r_f}\right)}{2\left(\frac{\varepsilon_2}{r_f}\right)}\right] - \frac{\sigma_{xx_{V}}}{2}$$

$$A_K = \left[\sigma_{xx_{I}} + \frac{2}{\sqrt{3}}\sigma_o\left(\frac{m}{\sin 2\alpha}\right)\right]\left\{\left[\frac{\left\langle\left(\frac{r_o}{r_f}\right) - 2\left(\frac{r}{r_f}\right)\right\rangle\left\langle -6 - 3\left(\frac{\varepsilon_1}{r_f}\right)\right\rangle}{8\left(\frac{\varepsilon_2}{r_f}\right)\left\langle\left(\frac{r_o}{r_f}\right) - \left(\frac{\varepsilon_2}{r_f}\right)\right\rangle}\right] + \right. \tag{90}$$

$$\left[\frac{1 + \left(\frac{\varepsilon_1}{r_f}\right)}{\left(\frac{\varepsilon_1}{r_f}\right)}\right] \cdot \left[\frac{\left(\frac{r_o}{r_f}\right) - 2\left(\frac{r}{r_f}\right)}{\left(\frac{\varepsilon_2}{r_f}\right)}\right] \ln\left(1 + \frac{\varepsilon_1}{r_f}\right) - \left[\sigma_{xx_{V}} + \frac{2}{\sqrt{3}}\sigma_o\left(\frac{m}{\sin 2\alpha}\right)\right] \cdot$$

$$\left[\frac{\left(\frac{r_o}{r_f}\right) - 2\left(\frac{r}{r_f}\right)}{2\left(\frac{\varepsilon_2}{r_f}\right)}\right] + \frac{\sigma_{xx_{V}}}{2} - \frac{2}{\sqrt{3}}\sigma_o\left[1 + \frac{m}{\sin 2\alpha}\right] \cdot$$

$$\left[\frac{\left(\frac{r_o}{r_f}\right) - 2\left(\frac{r}{r_f}\right)}{\left(\varepsilon_2/r_f\right)}\right] \ln\left\langle\left(\frac{r_o}{r_f}\right) - \left(\frac{\varepsilon_2}{r_f}\right)\right\rangle .$$

The stress ($\sigma_{\theta\theta}^{IV}$) is found by substituting equation (90) into equation (86) and then integrating the function with respect to (θ). (7)

$$\sigma_{\theta\theta}^{IV} = \frac{1}{e^{\theta}}\left\{\left[\frac{1}{5}A_s + \frac{2}{5}A_c\right]e^{\theta}\sin 2\theta \right.$$

$$\left. + \left[\frac{1}{5}A_c - \frac{2}{5}A_s\right]e^{\theta}\cos 2\theta + A_K e^{\theta} + f^{IV}(r)\right\} \tag{91}$$

The integration constant $\left[f^{IV}(r)\right]$ is determined by satisfying the yield criterion [equation (38)] at ($\theta = 0^o$). Evaluating (σ_{rr}^{IV}) in equation (84), ($\sigma_{\theta\theta}^{IV}$) in equation (91) and ($\sigma_{r\theta}^{IV}$) in equation (82), all at ($\theta = 0^o$), and then substituting them into equation (38) gives

$$f^{IV}(r) = B_c + B_K - \left[\frac{1}{5}A_c - \frac{2}{5}A_s\right] - A_K - \frac{2}{\sqrt{3}}\sigma_o ,$$

where

$$B_c = \left[\sigma_{xx_{I}} + \frac{2}{\sqrt{3}}\sigma_o\left(\frac{m}{\sin 2\alpha}\right)\right]\left[\frac{\left\langle\left(\frac{r_o}{r_f}\right) - \left(\frac{r}{r_f}\right)\right\rangle\left\langle 6 + 3\left(\frac{\varepsilon_1}{r_f}\right)\right\rangle}{8\left(\frac{\varepsilon_2}{r_f}\right)\left\langle\left(\frac{r_o}{r_f}\right) - \left(\frac{\varepsilon_2}{r_f}\right)\right\rangle}\right] \tag{92}$$

$$-\left[\sigma_{xx_{V}} + \frac{2}{\sqrt{3}}\sigma_o\left(\frac{m}{\sin 2\alpha}\right)\right]\left[\frac{\left(\frac{r_o}{r_f}\right) - \left(\frac{r}{r_f}\right)}{2\left(\varepsilon_2/r_f\right)}\right] + \frac{1}{2}\sigma_{xx_{V}}$$

$$B_K = \left[\sigma_{xx_I} + \frac{2}{\sqrt{3}}\sigma_0\left(\frac{m}{\sin 2\alpha}\right)\right]\left\{\left[\frac{\left\langle\left(\frac{r_0}{r_f}\right)-\left(\frac{r}{r_f}\right)\right\rangle\left\langle -6-3\left(\frac{\varepsilon_1}{r_f}\right)\right\rangle}{8\left(\frac{\varepsilon_2}{r_f}\right)\left\langle\left(\frac{r_0}{r_f}\right)-\left(\frac{\varepsilon_2}{r_f}\right)\right\rangle}\right] + \left[1+\frac{1}{\left(\frac{\varepsilon_1}{r_f}\right)}\right]\cdot\right.$$

$$\left.\left[\frac{\left(\frac{r_0}{r_f}\right)-\left(\frac{r}{r_f}\right)}{\left(\varepsilon_2/r_f\right)}\right]\ln\left[1+\left(\frac{\varepsilon_1}{r_f}\right)\right]\right\} - \left[\sigma_{xx_V} + \frac{2}{\sqrt{3}}\sigma_0\left(\frac{m}{\sin 2\alpha}\right)\right]\left[\frac{\left(\frac{r_0}{r_f}\right)-\left(\frac{r}{r_f}\right)}{2\left(\frac{\varepsilon_2}{r_f}\right)}\right]$$

$$+\frac{1}{2}\sigma_{xx_V} - \frac{2}{\sqrt{3}}\sigma_0\left[1+\left(\frac{m}{\sin 2\alpha}\right)\right]\left[\frac{\left(\frac{r_0}{r_f}\right)-\left(\frac{r}{r_f}\right)}{\left(\frac{\varepsilon_2}{r_f}\right)}\right]\ln\left[\left(\frac{r_0}{r_f}\right)-\left(\frac{\varepsilon_2}{r_f}\right)\right].$$

Therefore, after equation (92) is substituted into equation (91), $(\sigma_{\theta\theta}^{IV})$ becomes

$$\sigma_{\theta\theta}^{IV} = \frac{1}{e^{\theta}}\left\{\left[\frac{1}{5}A_s + \frac{2}{5}A_c\right]e^{\theta}\sin 2\theta + \left[\frac{1}{5}A_c - \frac{2}{5}A_s\right]\cdot\right.$$

$$\left.\left[e^{\theta}\cos 2\theta - 1\right] + A_K\left(e^{\theta}-1\right) + \left(B_c + B_K\right) - \frac{2}{\sqrt{3}}\sigma_0\right\} \qquad (93)$$

The stresses (σ_{rr}^{IV}), $(\sigma_{\theta\theta}^{IV})$ and $(\sigma_{r\theta}^{IV})$ are represented by smooth functions, and they obey the equilibrium equations and the imposed boundary conditions. The satisfaction of the von Mises yield criterion is now considered for the stress fields in zones II and III as well as in zone IV.

The entire stress fields in zones II, III and IV are now known. Each satisfies the continuity requirement, the equilibrium equations, the boundary conditions, and the von Mises yield criterion at $(\theta = 0^{\circ})$. However, these stress fields are not statically admissible until the von Mises yield criterion is satisfied throughout each zone. It is realized that the yield criterion varies with the (r) and the (θ) coordinates because the stresses (σ_{rr}), $(\sigma_{\theta\theta})$ and $(\sigma_{r\theta})$ are functions of (r) and (θ). To evaluate the yield criterion at many points in each zone, however, would be quite cumbersome. Therefore, the assumption is made that the von Mises yield criterion is obeyed throughout each zone if it is obeyed at the four corners of each zone. For example, the yield criterion must be satisfied at the following points in zone II: ($r = r_f$, $\theta = 0^\circ$), ($r = r_f$, $\theta = \alpha^\circ$), ($r = r_f + \varepsilon_1$, $\theta = 0^\circ$) and ($r = r_f + \varepsilon_1$, $\theta = \alpha^\circ$). Using the fact that the yield criterion must be satisfied throughout each zone allows a maximum lower bound value of (σ_{xxI}/σ_o) to be found without determining a formula relating (σ_{xxI}/σ_o) to the process variables.

The lower bound solution is not practical unless it is bounded by the "ideal" solution and an upper bound solution. These two solutions have been determined by Avitzur, et al. (3) for the problem of strip drawing under plane strain conditions. In their analysis, the "ideal" power of deformation ($\dot{W}_i$) was found to be

$$\dot{W}_i = \frac{2}{\sqrt{3}} \sigma_o v_f t_f \ln\left(\frac{t_o}{t_f}\right). \tag{94}$$

Since the external power is equal to the product of (t_f), (v_f) and (σ_{xxI}), equating ($\dot{W}_i$) to the external power gives

$$t_f v_f \sigma_{xxI} = \frac{2}{\sqrt{3}} \sigma_o v_f t_f \ln\left(\frac{t_o}{t_f}\right). \tag{95}$$

Therefore, the "ideal" drawing stress is

$$\frac{\sigma_{xxI}}{\sigma_o} = \frac{2}{\sqrt{3}} \ln\left(\frac{t_o}{t_f}\right). \tag{96}$$

The "ideal" drawing stress can be used as an initial point in determining the largest drawing stress (σ_{xxI}/σ_o) which still allows the yield criterion to be satisfied in each zone.

The procedure for determining the maximum drawing stress is as follows:

(1) Choose a set of typical process variables: ($\sigma_{xx_{I}}/\sigma_o$), (α), (t_o/t_f) and (m).

(2) As an initial point, allow

$$\left(\frac{\varepsilon_1}{r_f}\right) = \left(\frac{\varepsilon_2}{r_f}\right) = \left(\frac{\varepsilon}{r_f}\right) \tag{97}$$

and

$$r_f + \varepsilon_1 = r_o - \varepsilon_2 . \tag{98}$$

In other words, zone III would disappear, and

$$\left(\frac{\varepsilon}{r_f}\right) = \frac{1}{2}\left(\frac{t_o}{t_f} - 1\right) . \tag{99}$$

(Note that because of geometry, (r_o / r_f) = (t_o/t_f).)

(3) Substitute the process variables, the "ideal" value of (σ_{xxI}/σ_o) from equation (96), and the values for (ε_1/r_f) and (ε_2/r_f) into each of the stresses (σ_{rr}), ($\sigma_{\theta\theta}$) and ($\sigma_{r\theta}$) for each zone, and then evaluate the yield criterion for each zone.

(4) If the von Mises yield criterion is satisfied throughout zones II, III and IV, choose a larger value of the drawing stress (it must be less than the upper bound value (3).),

and then determine if the yield criterion is still obeyed. This step is repeated until a maximum value of the drawing stress is found for the chosen (ε_1 / r_f), (ε_2 / r_f) and process variables. In other words, the maximum (σ_{xx_I} / σ_o) is obtained when the yield criterion is still obeyed, but if another increment is added to (σ_{xx_I} / σ_o), the yield criterion will no longer be satisfied.

(5) Choose different values of (ε_1 / r_f) and ($\frac{\varepsilon_2}{r_f}$), keeping the process variables the same, and then repeat steps (3) and (4). (Remember that

$$r_f + \varepsilon_1 \leq r_o - \varepsilon_2 \tag{100}$$

or

$$\frac{\varepsilon_1}{r_f} \leq \frac{t_o}{t_f} - \frac{\varepsilon_2}{r_f} - 1 \ .)$$

Step (5) gives a maximum value of the drawing stress for a certain set of process variables and the values of ($\frac{\varepsilon_1}{r_f}$) and ($\frac{\varepsilon_2}{r_f}$) which maximize the drawing stress.

(6) Repeat the entire procedure for different process variables.

Obviously, this successive approximations procedure could be performed by the computer.

DISCUSSION AND RESULTS

The previously described method for determining the maximum drawing stress was worked for the process variables $(m = 0)$, $\left(\frac{\sigma_{xx_{I}}}{\frac{2}{\sqrt{3}}\sigma_0} = 0\right)$ and $\left(\frac{t_0}{t_f} = 1.2\right)$ as shown in Table I. (ε_1 / r_f) and (ε_2 / r_f) were both initially equal to 0.1 [equation (99)], and $\left(\frac{\sigma_{xx_I}}{\frac{2}{\sqrt{3}}\sigma_0}\right)$ was equal to .182, the "ideal" value found from equation (96). The von Mises yield criterion was evaluated first at the boundary $(r = r_f)$ and then solved for $(\theta = 0^\circ)$ and $(\theta = \alpha = 10^\circ)$ - the outermost points of zone II at $(r = r_f)$ when $(\alpha = 10^\circ)$. The yield criterion is obeyed at $(r = r_f)$ and $(\theta = 0^\circ)$ but not at $(r = r_f)$ and $(\theta = \alpha = 10^\circ)$ nor at any $(\theta = \alpha > 0^\circ)$.

Using the same process variables and the "ideal" drawing stress, the yield criterion was next evaluated for extreme values of (ε_1 / r_f). With a reduction of 1.2, equation (100) allows the maximum (ε_1 / r_f) to be equal to 0.2 and the minimum $\left(\frac{\varepsilon_1}{r_f}\right)$ to be zero. However, to keep the calculations simple, the values of (ε_1 / r_f) used were close to the maximum and minimum values. When (ε_1 / r_f) is equal to 0.198 or 0.002, the yield criterion is satisfied at $(r = r_f)$ and $(\theta = 0^\circ)$, but when $(\theta = \alpha > 0^\circ)$ and $(r = r_f)$, it is not obeyed.

These same calculations were made next for a drawing stress value equal to the upper bound solution value from reference (3),

Figure 9 (a), for 15 percent reduction in thickness $(t_o/t_f \approx 1.2)$ and $(\alpha = 10^\circ)$. Again the yield criterion is not obeyed. Although drawing stresses lower than the "ideal" value are not practical, $\left(\frac{\sigma_{xx_I}}{\frac{2}{\sqrt{3}}\sigma_o} = .14\right)$ was substituted into the yield criterion with $(\varepsilon_1/r_f = .1)$ to determine the location of this lower bound solution. However, this effort was of no avail either. From the results, if a lower bound value of the drawing stress exists for the process variables tested, it is below $\left(\frac{\sigma_{xx_I}}{\frac{2}{\sqrt{3}}\sigma_o} = .14\right)$.

The yield criterion is always satisfied at $(\theta = 0^\circ)$ and $(r = r_f)$. However, as can be seen in Table I, as (θ) becomes greater than zero, the left side of the inequality increases, and the right side decreases. This is opposite to what must occur for the satisfaction of the yield criterion at $(\theta = \alpha > 10^\circ)$. Therefore, if yielding is assumed at $(\theta = \alpha^\circ)$ instead of at $(\theta = 0^\circ)$ in order to solve for the stress fields, the possibility exists that the von Mises yield criterion could be satisfied for all values of (θ). Since assuming yielding at $(\theta = \alpha^\circ)$ instead of at $(\theta = 0^\circ)$ is likely to cause an extremely "bulky" solution, the method of solving for the stress fields without satisfying the von Mises yield criterion at some value of (θ) was considered first.

A different stress field in zone II was determined by assuming a distribution of the stress (σ_{rr}^{II}) at $(\theta = 0^\circ)$ instead of satisfying the von Mises yield criterion at $(\theta = 0^\circ)$. (The solution is found in Appendix II.) Several values of the drawing stress (equal to, less than and greater than the "ideal") were substituted into the stress field in an attempt to satisfy the yield criterion. The process variables and the values of $(\mathcal{E}_1 / r_f)$ and $(\mathcal{E}_2 / r_f)$ were the same as those used in the first approach.

As in the previous approach, the yield criterion is not obeyed at $(r = r_f)$ and $(\theta = \alpha > 0^\circ)$ with the "ideal" drawing stress or with the upper bound value from reference (3) substituted into the stresses in zone II [equations (26), (g) and (h)]. (Refer to Table II.) In fact, the stress field from this method does not even allow the yield criterion to be satisfied at $(r = r_f)$ and $(\theta = 0^\circ)$ while the stress field from the first approach does. However, at a value of $\left(\frac{\sigma_{xxI}}{\frac{2}{\sqrt{3}}\sigma_o} = .14\right)$, which is lower than the "ideal" drawing stress, the yield criterion is obeyed at the four corners of zone II: $(r = r_f)$ and $(\theta = 0^\circ)$, $(r = r_f)$ and $(\theta = \alpha = 10^\circ)$, $(r = r_f + \mathcal{E}_1 = r_o - \mathcal{E}_2)$ and $(\theta = 0^\circ)$, and $(r = r_f + \mathcal{E}_1 = r_o - \mathcal{E}_2)$ and $(\theta = \alpha = 10^\circ)$. $(\mathcal{E}_1 / r_f)$ was equal to 0.1. These calculations indicate that this approach gives a higher lower bound solution than the initial method. However, the second solution is not practical either because it is still lower than the "ideal" solution. Therefore, the stress fields in zones III and IV were not required, and the second approach was concluded.

At this point, one may wonder why the stress fields in zones III and IV were found in the first approach even though the von Mises yield criterion could not be satisfied in zone II. The reason is found by considering further the purpose of this project. Since a method of finding the stress fields in zones III and IV was not known initially, the first objective was actually to determine whether these stress fields could be established at all.

The results obtained from the calculations imply that neither of the lower bound solutions gives a drawing stress located between the "ideal" and upper bound drawing stresses found by Avitzur, et. al. (3). However, another factor [other than satisfying the von Mises yield criterion with assumed yielding at ($\theta = 0^\circ$)] which may have a significant effect on lowering the solution is that the drawing stress (σ_{xxI}) and the back pull (σ_{xxV}) are assumed constant throughout zones I and V, respectively. In other words, there is no variation of these stresses with any direction.

Assuming a constant stress in zones I and V allows the equilibrium equations to be satisfied along with the other requirements for a statically admissible stress field. However, the equilibrium equations and the other conditions could be obeyed in zones I and V even if the stress distribution in these zones is a function of the (x) and the (y) directions.

SUGGESTIONS FOR FUTURE WORK

From the results of this work, two approaches could be attempted next in order to determine a lower bound solution which is located between the "ideal" solution and the upper bound solution of Avitzur, et. al. (3).

(1) After assuming the constant statically admissible stress fields in zones I and V of Figure 1, determine deformation region stress fields which obey the imposed boundary conditions, the equations of equilibrium, and the von Mises yield criterion with assumed yielding at $(\Theta = \alpha^\circ)$. Then find the maximum drawing stress allowing the von Mises yield criterion to be satisfied throughout each deformation zone.

(2) Assume the stress distribution in zones I and V of Figure 1 to be represented by the following smooth functions obeying the equilibrium equations:

$$\text{Zone I;} \quad \sigma_{11} = \sigma_{xx_I} \cdot f(y), \quad \sigma_{22} = \sigma_{yy_I} \cdot f(x)$$

$$\text{Zone V;} \quad \sigma_{11} = \sigma_{xx_V} \cdot f(y), \quad \sigma_{22} = \sigma_{yy_V} \cdot f(x) \tag{101}$$

Then, along the entire die-wide strip interface, assume friction to obey the constant shear factor description. Therefore, the deformation region will contain only one zone. Next, solve for a statically admissible stress field in the deformation zone. (Note that ($\sigma_{r\theta}$) is assumed throughout this zone.) Finally, determining statically admissible stress fields in zones I and V enables the drawing force (P) to be found by integrating the components of ($\sigma_{rr} \big|_{r=r_f}$) and ($\sigma_{r\theta} \big|_{r=r_f}$) along the X-axis from ($\theta = 0°$) to ($\theta = \alpha°$).

CONCLUSIONS

(1) For the process of wide strip drawing in plane strain, stress fields in the deforming zones II, III and IV have been determined by using the approach described by Avitzur (2).

(2) Instead of satisfying the von Mises yield criterion with the assumption of yielding at ($\theta = 0^\circ$) as was done in the first approach, a different stress field was found in zone II by assuming a distribution of the stress (σ_{rr}^{II}) at ($\theta = 0^\circ$).

(3) Both approaches yield a lower bound solution located below the "ideal" solution of Avitzur, et. al. (3). However, the second method appears to give higher drawing stress values than does the first for the conditions listed in Tables I and II.

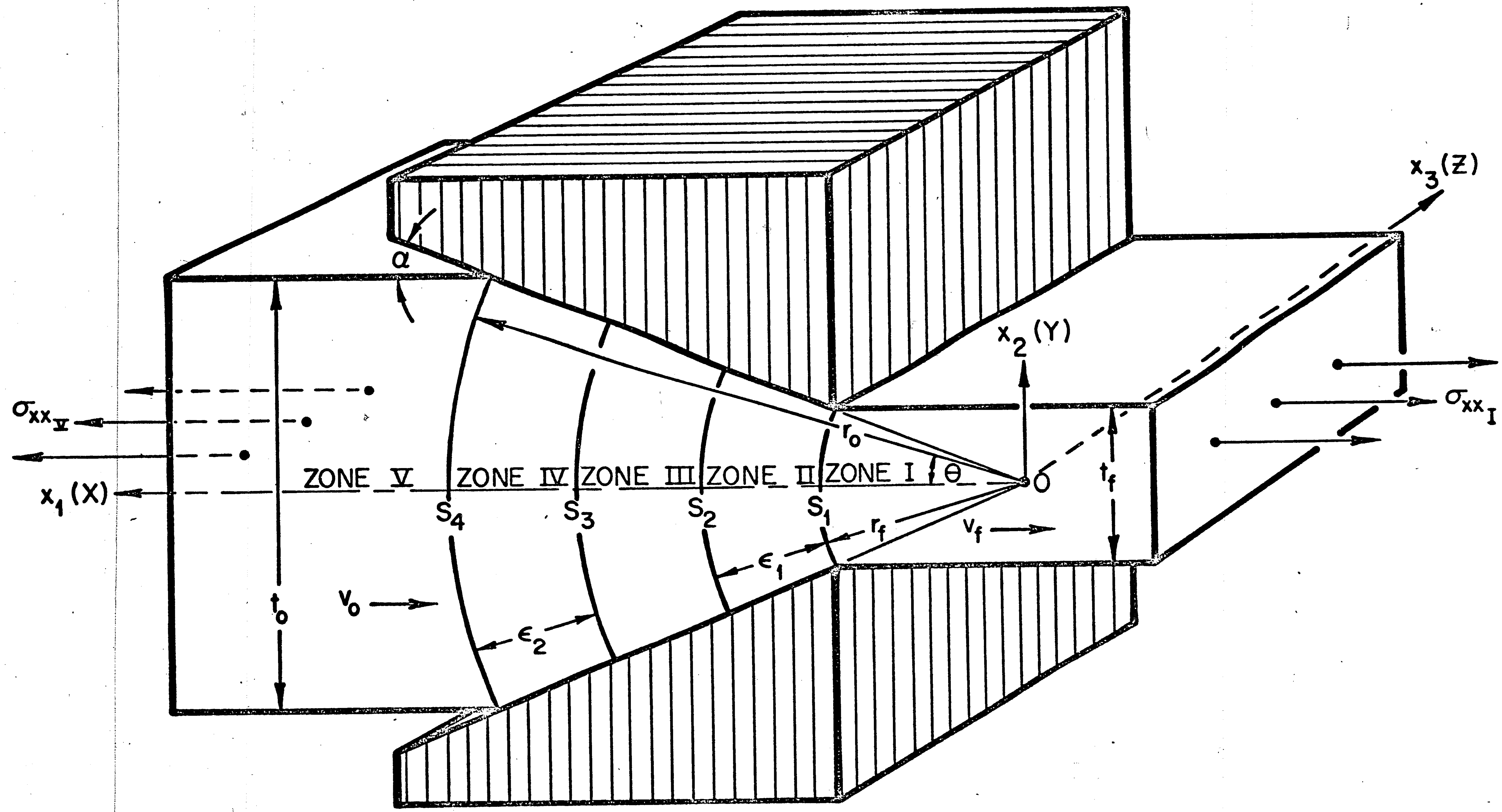

FIGURE 1 ASSUMED STATICALLY ADMISSIBLE STRESS FIELDS FOR WIDE STRIP DRAWING.

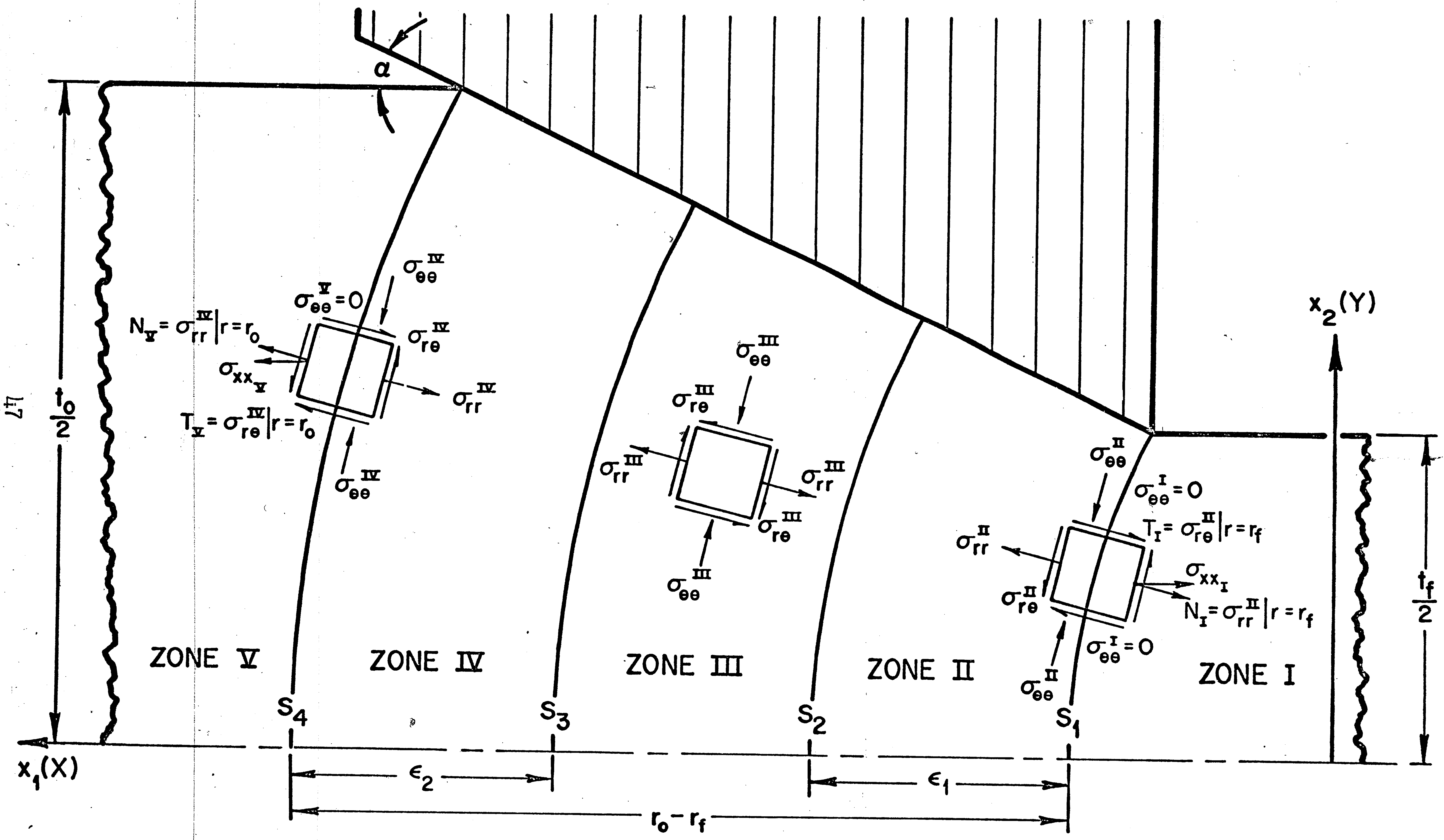

FIGURE 2 SURFACES OF STRESS DISCONTINUITY. (S)

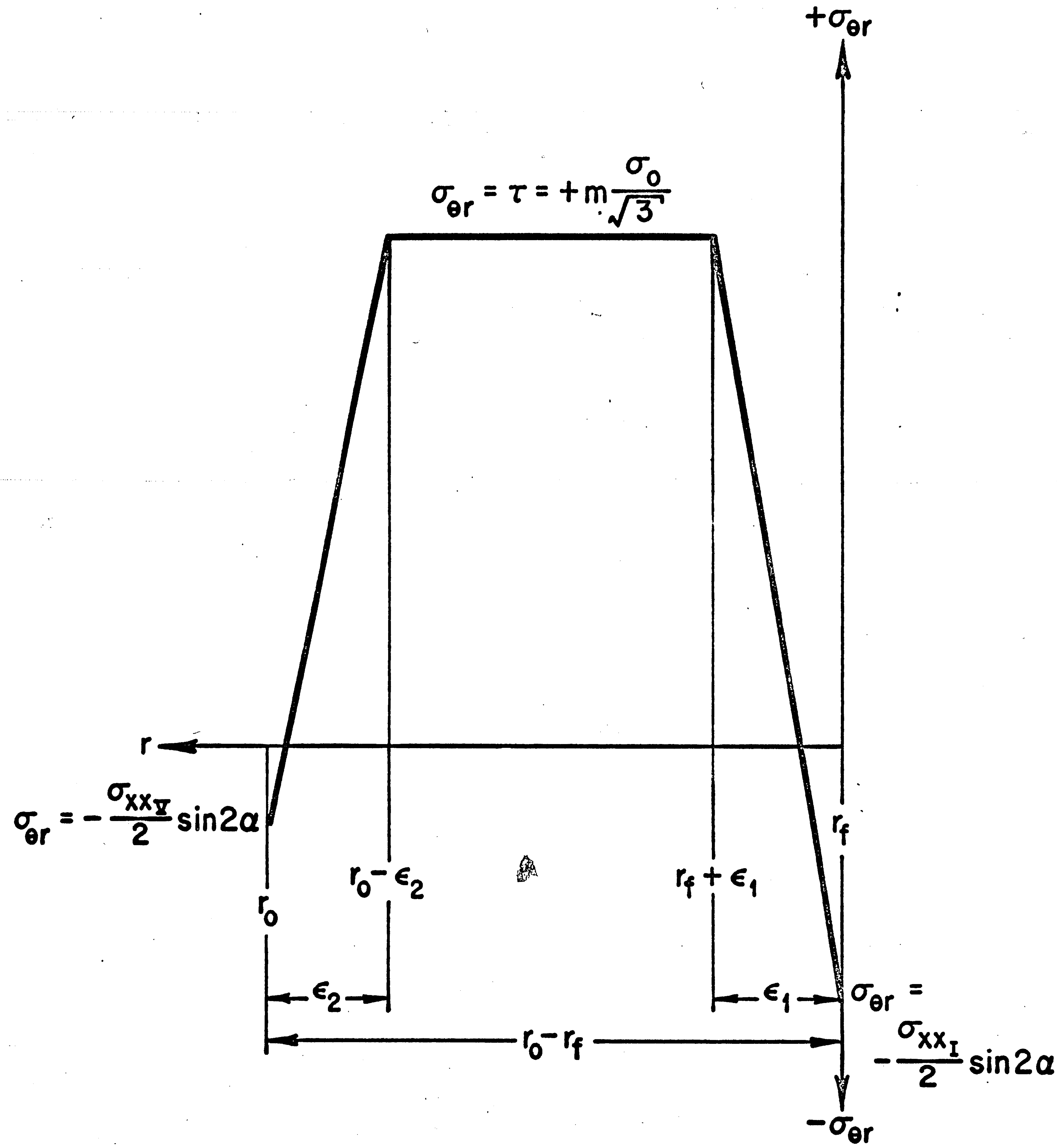

FIGURE 3 SHEAR STRESS ($\sigma_{\theta r}$) BEHAVIOR AT THE DIE-WIDE STRIP INTERFACE. ($\theta = \alpha°$)

TABLE I

EVALUATION OF THE von MISES YIELD CRITERION AT THE OUTERMOST POINTS OF THE DEFORMATION ZONES - FIRST APPROACH

PROCESS VARIABLES						Boundary of Evaluation	von MISES YIELD CRITERION [EQNS. (26), (48) and (49) $\sigma_{r\theta}$, σ_{rr} and $\sigma_{\theta\theta}$.] [EQN. (38)] - USED FOR	Θ	von MISES YIELD CRITERION AT Θ.
m	$\frac{\sigma_{xxV}}{\frac{2}{\sqrt{3}}\sigma_o}$	$\frac{t_o}{t_f}$	$\frac{\sigma_{xxI}}{\frac{2}{\sqrt{3}}\sigma_o}$	$\frac{\varepsilon_1}{r_f}$,	$\frac{\varepsilon_2}{r_f}$				
O	O	1.2	.182*	.1	.1	$r = r_f$	$1.273-.273\cos 2\theta \leq 2\sqrt{.250-(-.091\sin 2\theta)^2}$	0°	1 = 1 satisfied
O	O	1.2	.182*	.1	.1	$r = r_f$	$1.273-.273\cos 2\theta \leq 2\sqrt{.250-(-.091\sin 2\theta)^2}$	$\alpha = 10^\circ$	1.0166 > .998 not satisfied
O	O	1.2	.182*	.198	≤ .002	$r = r_f$	$1.048-.048\cos 2\theta \leq 2\sqrt{.250-(-.091\sin 2\theta)^2}$	$\alpha = 10^\circ$	1.003 > .998 not satisfied
O	O	1.2	.182*	.002	≤ .198	$r = r_f$	$23.618-22.618\cos 2\theta \leq 2\sqrt{.250-(-.091\sin 2\theta)^2}$	$\alpha = 10^\circ$	2.368 > .998 not satisfied
O	O	1.2	.260x	.1	.1	$r = r_f$	$1.390-.390\cos 2\theta \leq 2\sqrt{.250-(-.130\sin 2\theta)^2}$	$\alpha = 10^\circ$	1.024 > .996 not satisfied
O	O	1.2	.260x	.198	≤ .002	$r = r_f$	$1.068-.068\cos 2\theta \leq 2\sqrt{.250-(-.130\sin 2\theta)^2}$	$\alpha = 10^\circ$	1.004 > .996 not satisfied
O	O	1.2	.260x	.002	≤ .198	$r = r_f$	$33.240-32.240\cos 2\theta \leq 2\sqrt{.250-(-.130\sin 2\theta)^2}$	$\alpha = 10^\circ$	2.940 > .996 not satisfied
O	O	1.2	.140+	.1	.1	$r = r_f$	$1.210-.210\cos 2\theta \leq 2\sqrt{.250-(-.07\sin 2\theta)^2}$	$\alpha = 10^\circ$	1.013 > .9984 not satisfied

* "Ideal" value - Eqn. (96)

x Upper bound value for $\%r_t = 15$, $\alpha = 10^\circ$; [Fig. 9 (a), ref. (3)]

\+ Lower than "ideal" value

TABLE II

EVALUATION OF THE von MISES YIELD CRITERION AT THE OUTERMOST POINTS OF THE DEFORMATION ZONES - SECOND APPROACH

PROCESS VARIABLES									
m	$\frac{\sigma_{xxV}}{\frac{2}{\sqrt{3}}\sigma_o}$	$\frac{t_o}{t_f}$	$\frac{\sigma_{xxI}}{\frac{2}{\sqrt{3}}\sigma_o}$	$\frac{\varepsilon_1}{r_f}$	$\frac{\varepsilon_2}{r_f}$	Boundary of Evaluation	von MISES YIELD CRITERION [EQN. (38)] - [EQNS. (26) and (g) and (h) IN APPENDIX II USED FOR $\sigma_{r\theta}$, σ_{rr} and $\sigma_{\theta\theta}$.]	θ	von MISES YIELD CRITERION AT θ
0	0	1.2	.182*	.1	.1	$r = r_f$	$1.365-.273\cos 2\theta \leq 2\sqrt{.250-(-.091\sin 2\theta)^2}$	0°	1.092 > 1 not satisfied
0	0	1.2	.182*	.1	.1	$r = r_f$	$1.365-.273\cos 2\theta \leq 2\sqrt{.250-(-.091\sin 2\theta)^2}$	$\alpha = 10°$	1.108 > .998 not satisfied
0	0	1.2	.182*	.198	≤ .002	$r = r_f$	$1.140-.048\cos 2\theta \leq 2\sqrt{.250-(-.091\sin 2\theta)^2}$	$\alpha = 10°$	1.0949 > .998 not satisfied
0	0	1.2	.182*	.002	≤ .198	$r = r_f$	$23.710-22.518\cos 2\theta \leq 2\sqrt{.250-(-.091\sin 2\theta)^2}$	$\alpha = 10°$	2.560 > .998 not satisfied
0	0	1.2	.260x	.1	.1	$r = r_f$	$1.950-.390\cos 2\theta \leq 2\sqrt{.250-(-.130\sin 2\theta)^2}$	$\alpha = 10°$	1.583 > .996 not satisfied
0	0	1.2	.260x	.198	≤ .002	$r = r_f$	$1.628-.068\cos 2\theta \leq 2\sqrt{.250-(-.130\sin 2\theta)^2}$	$\alpha = 10°$	1.564 > .996 not satisfied
0	0	1.2	.260x	.002	≤ .198	$r = r_f$	$33.800-32.240\cos 2\theta \leq 2\sqrt{.250-(-.130\sin 2\theta)^2}$	$\alpha = 10°$	3.450 > .996 not satisfied
0	0	1.2	.140+	.1	.1	$r = r_f$	$1.05-.21\cos 2\theta \leq 2\sqrt{.250-(-.07\sin 2\theta)^2}$	0°	.840 < 1 satisfied
0	0	1.2	.140+	.1	.1	$r = r_f$	$1.05-.21\cos 2\theta \leq 2\sqrt{.250-(-.07\sin 2\theta)^2}$	$\alpha = 10°$	.853 < .9984 satisfied
0	0	1.2	.140+	.1	.1	$r = r_f + \varepsilon_1 = r_o - \varepsilon_2$	$1.0548-.2848\cos 2\theta \leq 2\sqrt{.250-\bar{\sigma}^2}$	0°	.770 < 1 satisfied
0	0	1.2	.140+	.1	.1	$r = r_f + \varepsilon_1 = r_o - \varepsilon_2$	$1.0548-.2848\cos 2\theta \leq 2\sqrt{.250-\bar{\sigma}^2}$	$\alpha = 10°$	.7868 < 1 satisfied

* "Ideal" value - EQN. (96)

x Upper bound value for $\%r_t = 15$, $\alpha = 10°$; [Fig. 9 (a), ref. (3)]

\+ Lower than "ideal" value

APPENDIX I

The Von Mises Yield Criterion for Plane Strain

In plane strain, from equation (20),

$$\sigma_{rz} = \sigma_{\theta z} = 0. \tag{a}$$

From von Mises' stress-strain rate law [equation (2)],

$$\dot{\varepsilon}_{zz} = \mu\, S_{zz}. \tag{b}$$

However, in plane strain,

$$\varepsilon_{zz} = 0; \quad \dot{\varepsilon}_{zz} = 0. \tag{c}$$

Therefore,

$$S_{zz} = 0. \tag{d}$$

Since

$$\sigma_{zz} = S + S_{zz} \tag{e}$$

and

$$S_{zz} = 0, \tag{d}$$

equation (e) reduces to

$$\sigma_{zz} = S. \tag{f}$$

The mean stress (S) is equal to the average of the three normal stresses.

$$S = \frac{1}{3}\left(\sigma_{rr} + \sigma_{\theta\theta} + \sigma_{zz}\right) \tag{g}$$

Combining equations (f) and (g) gives

$$\sigma_{zz} = \frac{1}{3}\left(\sigma_{rr} + \sigma_{\theta\theta} + \sigma_{zz}\right) \tag{h}$$

or

$$\sigma_{zz} = \frac{1}{2}\left(\sigma_{rr} + \sigma_{\theta\theta}\right)$$

Substituting equations (a) and (h) into equation (3) on page (6) yields the von Mises yield criterion for plane strain.

$$\frac{1}{6}\left[(\sigma_{rr}-\sigma_{\theta\theta})^2+\left(\sigma_{\theta\theta}-\frac{1}{2}(\sigma_{rr}+\sigma_{\theta\theta})\right)^2 + \left(\frac{1}{2}(\sigma_{rr}+\sigma_{\theta\theta})-\sigma_{rr}\right)^2\right]+\sigma_{r\theta}^2 \leq \frac{\sigma_0^2}{3} \tag{i}$$

This reduces to

$$(\sigma_{rr}-\sigma_{\theta\theta})^2 \leq 4\left(\frac{1}{3}\sigma_0^2-\sigma_{r\theta}^2\right)$$

$$\left|\sigma_{rr}-\sigma_{\theta\theta}\right| \leq 2\sqrt{\frac{1}{3}\sigma_0^2-\sigma_{r\theta}^2} \tag{j}$$

However, for drawing of wide strip in plane strain,

$$\begin{aligned} \sigma_{rr} &> 0 \; ; \quad \text{(tensile)} \\ \sigma_{\theta\theta} &< 0 \; ; \quad \text{(compressive)} . \end{aligned} \tag{k}$$

Therefore $(\sigma_{rr} - \sigma_{\theta\theta})$ is always positive, and the absolute value sign is not required.

$$\sigma_{rr}-\sigma_{\theta\theta} \leq 2\sqrt{\frac{1}{3}\sigma_0^2-\sigma_{r\theta}^2} \tag{38}$$

APPENDIX II

Determination of a Stress Field in Zone II by Assuming a Stress Distribution for (σ_{rr}) at $(\theta = 0^\circ)$

In this approach, the von Mises yield criterion, with the assumption of yielding at $(\theta = 0^\circ)$, is not satisfied in order to determine the stress field in zone II. Instead, a stress distribution for (σ_{rr}^{II}) is assumed at $(\theta = 0^\circ)$. This derivation is exactly the same as the first approach up to, and including, equations (36), and (37) for $(\sigma_{\theta\theta}^{II})$ and (σ_{rr}^{II}), respectively.

Equation (37) for (σ_{rr}^{II}) is

$$\sigma_{rr}^{II} = \left\{\left[+\frac{1}{2}\sigma_{xx_I} + \left(\frac{m}{\sin 2\alpha}\right)\frac{\sigma_0}{\sqrt{3}}\right]\left[\frac{r_f - \frac{1}{4}r}{\varepsilon_1}\right] + \frac{1}{2}\sigma_{xx_I}\right\}\cos 2\theta + F^{II}(r) + \left(\frac{1}{r}\right)f^{II}(\theta). \tag{a}$$

Using the boundary condition for (σ_{rr}^{II}) at $(r = r_f)$

$$\sigma_{rr}^{II}\big|_{r=r_f} = \sigma_{xx_I}\cos^2\theta \tag{45}$$

allows the function $[f^{II}(\theta)]$ to be found.

Substituting equation (45) into equation (a) evaluated at $(r = r_f)$ and then solving for $[f^{II}(\theta)]$ gives

$$f^{II}(\theta) = -r_f F^{II}(r_f) + r_f\sigma_{xx_I}\cos^2\theta - r_f\left\{\left[+\frac{1}{2}\sigma_{xx_I} + \left(\frac{m}{\sin 2\alpha}\right)\frac{\sigma_0}{\sqrt{3}}\right]\left(\frac{3}{4}\right)\left(\frac{r_f}{\varepsilon_1}\right) + \frac{1}{2}\sigma_{xx_I}\right\}\cos 2\theta. \tag{b}$$

Therefore, the stress (σ_{rr}^{II}) becomes

$$\sigma_{rr}^{II} = \left\{\left[+\frac{1}{2}\sigma_{xx_I} + \left(\frac{m}{\sin 2\alpha}\right)\frac{\sigma_0}{\sqrt{3}}\right]\left[\frac{r_f - \frac{1}{4}r}{\varepsilon_1}\right] + \frac{1}{2}\sigma_{xx_I}\right\}\cos 2\theta$$

$$+ F^{II}(r) - \left(\frac{r_f}{r}\right)F^{II}(r_f) + \left(\frac{r_f}{r}\right)\sigma_{xx_I}\cos^2\theta$$

$$-\left(\frac{r_f}{r}\right)\left\{\left[+\frac{1}{2}\sigma_{xx_I} + \left(\frac{m}{\sin 2\alpha}\right)\frac{\sigma_0}{\sqrt{3}}\right]\left(\frac{3}{4}\right)\left(\frac{r_f}{\varepsilon_1}\right) + \frac{1}{2}\sigma_{xx_I}\right\}\cos 2\theta. \qquad \text{(c)}$$

The unknown function [$F^{II}(r)$] in equation (c) can be found by assuming a distribution of the stress (σ_{rr}^{II}) at ($\theta = 0°$).

$$\sigma_{rr}^{II}\big/_{\theta=0°} = \sigma_{xx_I} + \left(\sigma_{xx_V} - \sigma_{xx_I}\right)\left(\frac{r - r_f}{r_0 - r_f}\right) \qquad \text{(d)}$$

Substituting equation (d) into equation (c) at ($\theta = 0°$) and then solving for [$F^{II}(r)$] yields

$$F^{II}(r) = \left(\frac{r_f}{r}\right)F^{II}(r_f) - \frac{1}{2}\left(\frac{r_f}{r}\right)\sigma_{xx_I} + \frac{1}{2}\sigma_{xx_I}$$

$$+ \left[\frac{1}{2}\sigma_{xx_I} + \left(\frac{m}{\sin 2\alpha}\right)\frac{\sigma_0}{\sqrt{3}}\right]\left[\left(\frac{3}{4}\right)\left(\frac{r_f}{r}\right)\left(\frac{r_f}{\varepsilon_1}\right) - \left(\frac{r_f - \frac{1}{4}r}{\varepsilon_1}\right)\right] \qquad \text{(e)}$$

$$+ \left(\sigma_{xx_V} - \sigma_{xx_I}\right)\left(\frac{r - r_f}{r_0 - r_f}\right).$$

Taking the derivative of [$F^{II}(r)$] [equation (e)] with respect to (r) and then multiplying by (r) gives the other unknown function needed to find ($\sigma_{\theta\theta}^{II}$) in equation (36).

$$r \cdot F'^{II}(r) = -\left(\frac{r_f}{r}\right)F^{II}(r_f) + \frac{1}{2}\left(\frac{r_f}{r}\right)\sigma_{xx_I}$$

$$+ \left[\frac{1}{2}\sigma_{xx_I} + \left(\frac{m}{\sin 2\alpha}\right)\frac{\sigma_0}{\sqrt{3}}\right]\left[-\left(\frac{3}{4}\right)\left(\frac{r_f}{r}\right)\left(\frac{r_f}{\varepsilon_1}\right) + \left(\frac{1}{4}\right)\left(\frac{r}{\varepsilon_1}\right)\right] \qquad \text{(f)}$$

$$+ \left(\sigma_{xx_V} - \sigma_{xx_I}\right)\left(\frac{r}{r_0 - r_f}\right)$$

The stress ratios ($\sigma_{rr}^{II}/\sigma_0$) and ($\sigma_{\theta\theta}^{II}/\sigma_0$) can now be determined.

($\frac{\sigma_{rr}^{II}}{\sigma_0}$) is found by substituting equation (e) into equation (c) and then dividing both sides of the resulting equation by (σ_0).

$$\frac{\sigma_{rr}^{II}}{\sigma_o} = \left[\frac{1}{2}\left(\frac{\sigma_{xx_I}}{\sigma_o}\right) + \frac{1}{\sqrt{3}}\left(\frac{m}{\sin 2\alpha}\right)\right]\left[\frac{4\left(\frac{r}{r_f}\right) - \left(\frac{r}{r_f}\right)^2 - 3}{4\left(\frac{r}{r_f}\right)\left(\frac{\varepsilon_1}{r_f}\right)}\right](\cos 2\theta - 1)$$

$$+\frac{1}{2}\left(\frac{\sigma_{xx_I}}{\sigma_o}\right)(\cos 2\theta + 1) + \left[\left(\frac{\sigma_{xx_{II}}}{\sigma_o}\right) - \left(\frac{\sigma_{xx_I}}{\sigma_o}\right)\right]\left[\frac{\left(\frac{r}{r_f}\right) - 1}{\left(\frac{t_o}{t_f}\right) - 1}\right] \qquad \text{(g)}$$

Substituting equations (e) and (f) into equation (36) and dividing by (σ_o) gives

$$\frac{\sigma_{\theta\theta}^{II}}{\sigma_o} = \left[+\frac{1}{2}\left(\frac{\sigma_{xx_I}}{\sigma_o}\right) + \frac{1}{\sqrt{3}}\left(\frac{m}{\sin 2\alpha}\right)\right]\left[\frac{\left(\frac{r}{r_f}\right) - 2}{2\left(\frac{\varepsilon_1}{r_f}\right)}\right](\cos 2\theta + 1)$$

$$+\left[+\frac{1}{2}\left(\frac{\sigma_{xx_I}}{\sigma_o}\right) + \frac{1}{\sqrt{3}}\left(\frac{m}{\sin 2\alpha}\right)\right]\left[\frac{\left(r/r_f\right)}{\left(\frac{\varepsilon_1}{r_f}\right)}\right]\cos 2\theta \qquad \text{(h)}$$

$$-\frac{1}{2}\left(\frac{\sigma_{xx_I}}{\sigma_o}\right)(\cos 2\theta - 1) + \left[\left(\sigma_{xx_{II}}/\sigma_o\right) - \left(\frac{\sigma_{xx_I}}{\sigma_o}\right)\right]\left[\frac{2\left(\frac{r}{r_f}\right) - 1}{\left(t_o/t_f\right) - 1}\right].$$

The stress ratio ($\sigma_{zz}^{II}/\sigma_o$) is found from equation (h) in Appendix I if required.

REFERENCES

1. Alexander, J.M., "On Complete Solutions for Frictionless Extrusion in Plane Strain," Q. Applied Mathematics, v. 19, no. 32, 1961, pp. 31-37.

2. Avitzur, B., Metal Forming--Processes and Analysis, McGraw-Hill Book Company, New York, N.Y., 1968.

3. Avitzur, B., Fueyo, J., and Thompson, J., "Analysis of Plastic Flow Through Inclined Planes in Plane Strain", ASME Paper No. 66-WA/Prod-22, presented at the Winter Annual Meeting, New York, N.Y., November 27-December 1, 1966, of The American Society of Mechanical Engineers.

4. Byars, E.F. and Snyder, R.D., Engineering Mechanics of Deformable Bodies, International Textbook Co., Scranton, Pa., 1963.

5. Devenpeck, M.L. and Richmond, O., "Strip Drawing Experiments with a Sigmoidal Die Profile," Journal of Engineering for Industry, Trans. ASME, Series B, vol. 87, 1965, pp. 425-428.

6. Dieter, G.E., Jr., Mechanical Metallurgy, McGraw-Hill Book Co., Inc., New York, 1961, pp. 73-77.

7. Dwight, H.B., Tables of Integrals and Other Mathematical Data, fourth edition, The Macmillan Co., New York, 1965, integral numbers 577.1 and 577.2.

8. Ellis, F., "Stress Discontinuities in Plane Plastic Flow", Journal of Strain Analysis, v. 2, No. 1, 1967, pp. 52-61.

9. Evans, W. and Avitzur, B., "An Experimental Study of Friction, Distortion and Flow Patterns in Drawing and Hydrostatic Extrusion", Report to the AISI, April, 1967.

10. Fukui, S., Ohi, T., Kudo, H., Takita, I., and Seino, J., "Some Aspects of Friction in Metal-Strip Drawing", International Journal of Mechanical Sciences, Pergamon Press Ltd., vol. 4, 1962, pp. 297-312.

11. Golomb, M. and Shanks, M., Elements of Ordinary Differential Equations, second edition, McGraw-Hill Book Co., 1965, pp. 39-42.

12. Halling, J. and Mitchell, L.A., "Use of Upper Bound Solutions for Predicting the Pressure for the Plane Strain Extrusion of Materials", J. Mech. Eng. Sci., vol. 6, no. 3, 1964, pp. 240-249.

13. Hill, R., The Mathematical Theory of Plasticity, Clarendon Press, Oxford, 1950, p. 128.

14. Hill, R. and Tupper, S.J., "A New Theory of the Plastic Deformation in Wire-Drawing", Journal of Iron and Steel Institute, vol. 159, 1948, pp. 353-359.

15. Lancaster, P. R. and Rowe, G.W., "Experimental Study of the Influence of Lubrication Upon Cold Drawing Under Approximately Plane-Strain Conditions at Low Speed", Proc., The Institution of Mechanical Engineers, vol. 178, part 1, no. 3, 1963-1964, pp. 69-89.

16. von Mises, R., "Mechanik der Festen Koerper im plastisch deformablen Zustand", Goettinger Nachrichten, mathematisch-physikalische Klasse, 1913, pp. 582-592.

17. Prager, W. and Hodge, P.G. Jr., Theory of Perfectly Plastic Solids, John Wiley and Sons, Inc., New York, N.Y., 1951.

18. Richmond, O. and Devenpeck, M.C., "A Die Profile for Maximum Efficiency in Strip Drawing", Proc. of the Fourth U.S. National Congress of Applied Mechanics, vol. 2, 1962, pp. 1053-1057.

19. Shield, R.T., "Plastic Flow in a Converging Conical Channel", Journal of the Mechanics and Physics of Solids, v. 3, 1955, pp. 246-258.

20. Sokolnikoff, I.S., Mathematical Theory of Elasticity, second edition, McGraw-Hill Book Company, New York, N.Y., 1956.

21. Unksov, E.P., An Engineering Theory of Plasticity, translated from Russian by Production Engineering Research Association of Great Britain, London, England, Butterworth, 1961, p. 78.

VITA

Fred R. Sauerwine was born on November 19, 1944, in Huntingdon, Pennsylvania, to Mary K. and George H. Sauerwine.

While attending Pen Argyl Area Joint High School, he was elected to the National Honor Society of Secondary Schools. He graduated in June, 1962.

In September, 1962, he entered Lafayette College to study metallurgical engineering. In his senior year, he was elected into the membership of Alpha Sigma Mu. He also received an award from the Lehigh Valley chapter of the American Society for Metals. In June, 1966, he was awarded a Bachelor of Science degree in Metallurgical Engineering.

On January 26, 1963, he married the former Bonnie L. Hooper. They now have a son, Scott Alan.